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Ministry of Education, Higher Education and Vocational Education

Support Seminar – 2025

10 – Combined Mathematics

Paper I

Marking Scheme

1. Using the **Principle of Mathematical Induction**, prove that

$$1.3 + 2.3^2 + 3.3^3 + \cdots\cdots + n.3^n = \frac{(2n-1)(3)^{n+1}+3}{4} \text{ for all } n\epsilon\mathbb{Z}^+.$$

For $n = 1$; $L.H.S = 1.3$

$R.H.S = \frac{(2.1-1)(3)^{1+1}+3}{4} = \frac{(2-1)\,9+3}{4} = \frac{12}{4} = 3$

$L.H.S = R.H.S$ $\therefore$ *the statement is true for* $n = 1$. (5)

Assume that the statement is true for $n = k$; $k\epsilon\mathbb{Z}^+$

$n = k$; $1.3 + 2.3^2 + 3.3^3 + \cdots\cdots + k.3^k = \frac{(2k-1)(3)^{k+1}+3}{4}$ ——— (1) (5)

$n = k + 1$ *from* (1)

$$1.3 + 2.3^2 + 3.3^3 + \cdots\cdots + k.3^k + (k+1).3^{(k+1)} = \frac{(2k-1)(3)^{k+1}+3}{4} + (k+1).3^{(k+1)}$$ (5)

$$= \frac{(3)^{k+1}\,[2k-1+4(k+1)]+3}{4}$$

$$= \frac{(3)^{k+1}\,[6k+3]+3}{4}$$

$$= \frac{(3)^{k+1}.3^1[2k+1]+3}{4}$$

$$= \frac{(3)^{k+2}.[2(k+1)-1]+3}{4}$$ (5)

$$= \frac{(3)^{(k+1)+1}.[2(k+1)-1]+3}{4}$$

Hence, if the statement is true for $n = k$, *then for* $n = k + 1$ *is also true.*
By the principle of mathematical induction, the statement is true for all $n\epsilon\mathbb{Z}^+$. (5)

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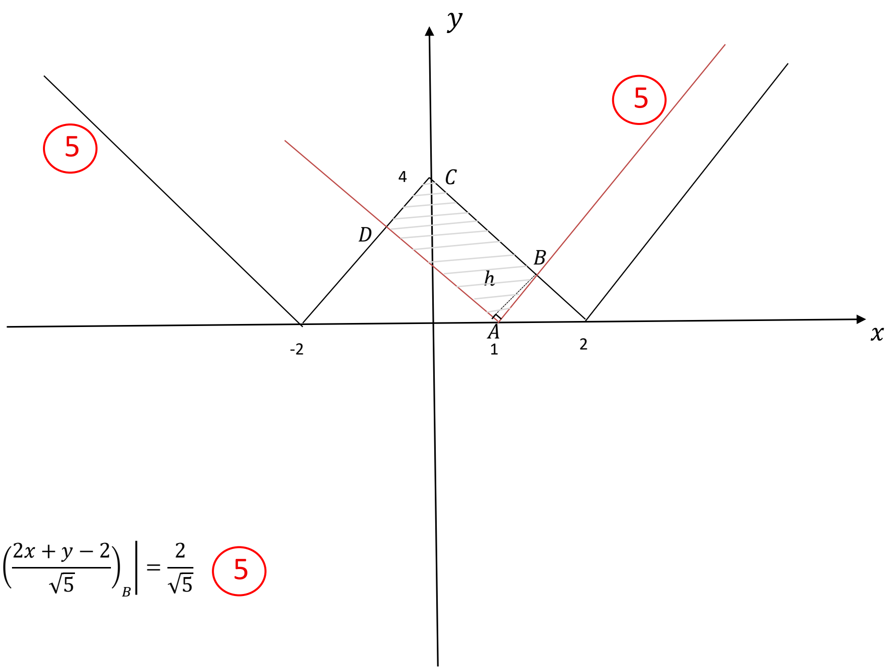
2. Sketch the graphs of $y = \left|4 - 2|x|\right|$ and $y = 2|x - 1|$ in the same diagram where $x \in \mathbb{R}$. **Hence,** find the area of the region enclosed by these graphs.



$A \equiv (1,0)$

$B \equiv (\frac{3}{2}, 1)$

$C \equiv (0,4)$

$D \equiv (-\frac{1}{2}, 3)$

$AD \equiv 2x + y - 2 = 0$; $h = \left|\left(\frac{2x+y-2}{\sqrt{5}}\right)_B\right| = \frac{2}{\sqrt{5}}$ (5)

$d(AD) = \frac{3\sqrt{5}}{2}$ (5)

$Area\ of\ ABCD = OC \times h = 3$ (5)

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3 Let z, w are two complex numbers such that $|z - 2| = |w + 2| = 1$, then prove that $\frac{-\pi}{6} \leq Arg\,(z - w) \leq \frac{\pi}{6}$. Find the range of $|z - w|$.

$Arg(z - w) = \theta, |z - w| = PQ$

$\Delta OAC \rightarrow \sin\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$ (5)

$Arg(z - w)_{max} = \alpha$ *when P is at C and Q is at F*

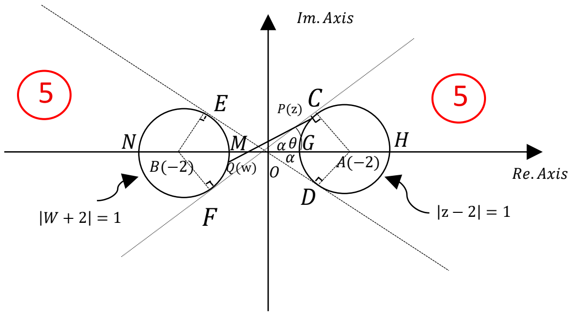
$Arg(z - w)_{min} = -\alpha$ *when P is at D and Q is at E*

$\therefore \quad -\frac{\pi}{6} \leq Arg(z - w) = \frac{\pi}{6}$ (5)

$PQ_{max} = NH = 6$

$PQ_{min} = MG = 2$

$\therefore \quad 2 \leq |z - w| \leq 6$ (5)



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4. Write down the binomial expansion of $\left(x + \frac{1}{x}\right)^n$ in ascending power of x for $n \in \mathbb{Z}^+$. **Hence,** show that the coefficient of x^2 in the expansion of $\left(\left(x + \frac{1}{x}\right)^5 + x\right)^2$ is 231.

Answer:

$\left(x + \frac{1}{x}\right)^n = \sum_{r=0}^{n} n_{C_r} x^r \left(\frac{1}{x}\right)^{n-r}$ (5) ; where $n_{C_r} = \frac{n!}{r!(n-r)!}$ *for* $r = 1,2,\ldots n$ and $n_{C_0} = 1$ (5)

$\left(\left(x + \frac{1}{x}\right)^5 + x\right)^2 = \left(x + \frac{1}{x}\right)^{10} + 2x\left(x + \frac{1}{x}\right)^5 + x^2$

General term of $\left(x + \frac{1}{x}\right)^{10} = 10_{C_r}.\,x^r\left(\frac{1}{x}\right)^{10-r} = 10_{C_r}x^{2r-10}$

General term of $2x\left(x + \frac{1}{x}\right)^5 = 2 \times x \times 5_{C_r}.\,x^r\left(\frac{1}{x}\right)^{5-r} = 2 \times 5_{C_r}.\,x^{2r-4}$

Cases for x^2 term,

$10_{C_r}x^{2r-10}$	$2 \times 5_{C_r}.\,x^{2r-4}$	x^2
$2r - 10 = 2$	$2r - 4 = 2$	
$r = 6$ (5)	$r = 3$ (5)	
10_{C_6}	$2 \times 5_{C_3}$	1

$$\frac{10!}{6! \times 4!} + 2 \times \frac{5!}{2! \times 3!} + 1$$

$$\left(\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}\right) + 2 \times \left(\frac{5 \times 4}{2 \times 1}\right) + 1 = 231 \quad (5)$$

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5. Show that $\lim_{x\to\frac{\pi}{6}}\left(\frac{2-\sqrt[3]{16\sin x}}{\sin 6x}\right)=\frac{1}{3\sqrt{3}}$

$$L.H.S=(-1)\lim_{n\to\frac{\pi}{6}}\left(\frac{(16\sin x)^{\frac{1}{3}}-8^{\frac{1}{3}}}{16\sin x-8}\right).\frac{(\pi-6x)}{\sin(\pi-6x)}.\frac{16\left(\sin x-\sin\frac{\pi}{6}\right)}{(\pi-6x)}$$ 5

5

$$=(-16).\lim_{16\sin x\to 8}\left(\frac{(16\sin x)^{\frac{1}{3}}-8^{\frac{1}{3}}}{16\sin x-8}\right).\lim_{(\pi-6x)\to 0}\left(\frac{(\pi-6x)}{\sin(\pi-6x)}\right).\lim_{\left(\frac{x-\frac{\pi}{6}}{2}\right)\to 0}\left[\frac{2\cos\left(x+\frac{\pi}{6}\right)}{-12}.\left(\frac{\sin\left(\frac{x-\frac{\pi}{6}}{2}\right)}{\left(\frac{x-\frac{\pi}{6}}{2}\right)}\right)\right]$$

5

$$=-16.\frac{1}{3}8\times\frac{1}{(1)}\times\left(\frac{-1}{6}\right).\frac{\sqrt{3}}{2}.(1)$$

5 5

$$=\frac{1}{3\sqrt{3}}$$

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6. It is given that when $y=(a+x^2)\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)$; $a>0$, $\frac{dy}{dx}=\sqrt{a}+2x\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)$.

Let $f(x=\sqrt{x\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)}$ for $a,x>0$. Show that the volume of the solid generated, when the region bounded by the curve $y=f(x)$, the x –axis, $x=0$ and $x=\sqrt{a}$, is rotated about the x-axis by 2π radians is $\pi\left[\frac{a}{4}(\pi-2)\right]$ cubic units.

$$volume,\quad V=\pi\int_0^{\sqrt{a}}[f(x)]^2dx$$

$$=\pi\int_0^{\sqrt{a}}x\tan^{-1}\left(\frac{x}{\sqrt{x}}\right)dx$$ 5

$$\frac{dy}{dx}=\sqrt{a}+2x\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)$$

$$\therefore\int_0^{\sqrt{a}}\left(\frac{dy}{dx}\right)dx=\int_0^{\sqrt{a}}\sqrt{a}dx+\int_0^{\sqrt{a}}2x\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)dx$$ 5

$$\int_0^{\sqrt{a}}x\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)dx=\frac{1}{2}[y]_o^{\sqrt{a}}-\frac{1}{2}\sqrt{a}\int_0^{\sqrt{a}}dx$$ 5

$$=\frac{1}{2}\left[(a+x^2)\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)-\sqrt{a}x\right]_0^{\sqrt{a}}$$

$$=\frac{1}{2}\left[\left((a+a)\tan^{-1}\left(\frac{\sqrt{a}}{\sqrt{a}}\right)-\sqrt{a}\sqrt{a}\right)-(0-0)\right]$$ 5

$$=\frac{1}{2}\left(2a\frac{\pi}{4}-a\right)$$

$$=\frac{a}{2}\left(\frac{\pi}{2}-1\right)$$

$$Volume=(\pi)\left[\frac{a}{4}(\pi-2)\right]cubic\ units\ cubic\ units$$ 5

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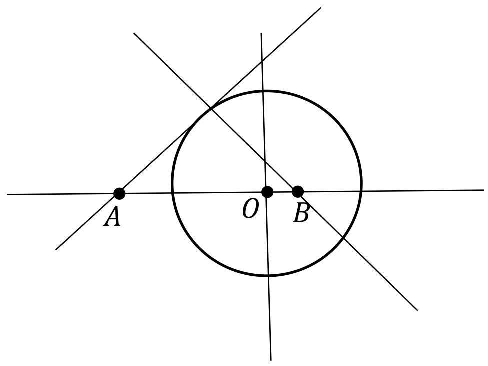
7. An ellipse is given by parametric equation $x = 13\cos\alpha$, $y = 5\sin\alpha$. Where, α is a parameter. Show that $\frac{dy}{dx} = -\frac{5}{13}\cot\alpha$ for $\alpha \neq n\pi, \alpha \neq (2n+1)\frac{\pi}{2}$, $n \epsilon \mathbb{Z}$. Write the equation of tangent and the normal drawn to the curve at the point of parameter α. Let these tangent and normal cut the x - axis at A and B respectively, show that $OA \,.\, OB = 144$ where O is the origin.

$x = 13\cos\alpha$ $\quad$ $y = 5\sin\alpha$

$\frac{d(x)}{d\alpha} = -13\sin\alpha$ $\quad$ $\frac{d(y)}{d\alpha} = 5\cos\alpha$ $\quad$ (5)



$\frac{dy}{dx} = \frac{-5\cos\alpha}{13\sin\alpha} = \frac{-5}{13}\cot\alpha$ (5)

$y - 5\sin\alpha = -\frac{5}{13}\cot\alpha\,(x - 13\cos\alpha)$

$y - 5\sin\alpha = \frac{13}{5}\tan\alpha\,(x - 13\cos\alpha)$ $\quad$ (5)

When $y = 0$ for

$x_1 = 13\sin\alpha \times \frac{\sin\alpha}{\cos\alpha} + 13\cos\alpha = \frac{13\sin^2\alpha + 13\cos^2\alpha}{\cos\alpha} = \frac{13}{\cos\alpha}, A \equiv \left(\frac{13}{\cos\alpha}, 0\right)$ (5)

$x_2 = -\frac{25}{13}\cos\alpha + 13\cos\alpha = \frac{144}{3}\cos\alpha$ $\quad$ $B \equiv \left(\frac{144}{13}\cos\alpha, 0\right)$

$OA.OB = \frac{13}{\cos\alpha} \times \frac{144}{13}\cos\alpha = 144$

(5)

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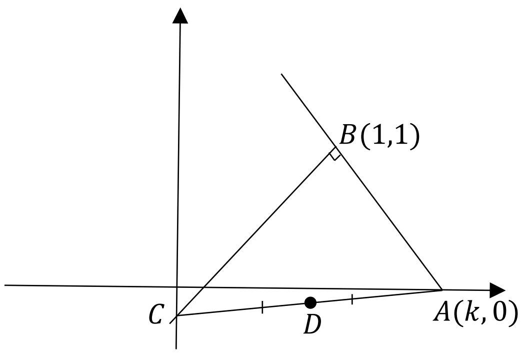
8. Let $A \equiv (k, 0)$ where $k \in \mathbb{R} - \{1\}$ and $B \equiv (1,1)$. The line passing through B, perpendicular to AB, cuts y –axis at C. Find the coordinates of the midpoint D of AC in terms of k and find the locus of D as k varies.



$m_{AB} = \frac{1-0}{1-k}$, $k \neq 1$

$\therefore m_{BC} = \frac{1}{k-1}$ (5)

$\therefore BC; y - 1 = \frac{1}{(k-1)}(x - 1)$ (5)

$y = (k-1)(x-1) + 1$

$y = (k-1)x + (2-k)$ (5)

$when\ x = 0, y = (2-k)$

$\therefore D = \left(\frac{0+k}{2}, \frac{2-k}{2}\right) = \left(\frac{k}{2}, \frac{2-k}{2}\right)$ (5)

$locus \implies x + y - 1 = 0$

(5)

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9. Let S be the circle with equation $x^2 + y^2 - 8x - 10y + 16 = 0$. Another circle S_1 of radius 4 units touches the y-axis at a point above the centre of S. The circles S and S_1 touch internally. Find the equation of S_1.

(5)

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\sqrt{g^2 + f^2 - c} = 4$ $\quad\quad$ $|-g| = 4$

$g < 0 \implies g = -4$ (5)

$16 + f^2 - c = 16$

$c = f^2$

$(4 + g)^2 + (5 + f)^2 = 5 - 4$ (5)

$0 + (5 + f)^2 = 1$

$-f > 5 \implies f = -6$

$c = 36$ (5)

$S_1 \equiv x^2 + y^2 - 8x - 12y + 36 = 0$ (5)

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10. Solve the equations $x + y = \frac{\pi}{4}$, $\tan x + \tan y = 1$

$\tan x + \tan y = 1$

$\frac{\sin x}{\cos y} + \frac{\sin y}{\cos y} = 1$

$\sin x \cos y + \cos x \sin y = \cos x \cos y$ (5)

$2\sin(x + y) = \cos(x + y) + \cos(x - y)$

$2\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + \cos(x - y)$

$\cos(x - y) = \frac{1}{\sqrt{2}}$ (5)

$x - y = 2n\pi \pm \frac{\pi}{4} \longrightarrow (1)$ (5)

$x + y = \frac{\pi}{4} \longrightarrow (2)$

$form\ (1) and\ (2)$ $\quad x = n\pi + \frac{\pi}{4}$ $\ or\ x = n\pi$ (5)

$y = -n\pi, n \in \mathbb{Z}, y = -n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$ (5)

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Ministry of Education, Higher Education and Vocational Education

G.C.E. (A.L) Support Seminar - 2025

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

✱ **Answer five questions only.**

11. (a) Let $f(x) = x^2 - \lambda x + c^2$, where λ and c are real constants. Show that the minimum value of $f(x)$ is $\frac{4c^2-\lambda^2}{4}$. If the graph of $y = f(x)$ intersects the x- axis at two distinct points, **deduce** that $\lambda > 2|c|$.

Let α and β be the roots of $f(x) = 0$. Given that both α and β are positive, show that $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{\lambda + 2|c|}$.

Hence, show that the quadratic equation whose roots are $\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$ and $\sqrt{\beta} - \frac{1}{\sqrt{\beta}}$ is given by $|c|x^2 - \sqrt{\lambda + 2|c|}(|c| - 1)x + c^2 - \lambda + 1 = 0$.

Let $g(x) = |c|x^2 - \sqrt{\lambda + 2|c|}(|c| - 1)x + c^2 - \lambda + 1$.

If the graphs of $y = f(x)$ and $y = g(x)$ intersect on the y- axis, show that $\lambda = 1$.

(b) Let $f(x)$ is a polynomial of degree 3. When $f(x)$ is divided by $x^2 - 3x + 2$, $x^2 - 5x + 6$ and $x^2 - 4x + 3$ separately, the remainder is $11x - 6$. If $f(0) = 0$, find $f(x)$.

Further **deduce** that maximum value of the gradient of $y = f(x)$ is 12.

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11. (a).

$$f(x) = x^2 - \lambda x + c^2$$

$$f(x) = \left(x - \frac{\lambda}{2}\right)^2 + \frac{4c^2-\lambda^2}{4}$$

$$\left(x - \frac{\lambda}{2}\right)^2 \geq 0 \quad (5)$$

$$\left(x - \frac{\lambda}{2}\right)^2 + c^2 - \frac{\lambda^2}{4} \geq \frac{4c^2-\lambda^2}{4}$$

$$f(x) \geq \frac{4c^2-\lambda^2}{4}$$

$$\therefore f(x) minimum = \frac{4c^2-\lambda^2}{4} \quad (5)$$

If graph of $y = f(x)$ intersects the x-axis at two distinct points, $\frac{4c^2-\lambda^2}{4} < 0$ (10)

$\therefore 4c^2 < \lambda^2$

$(2c - \lambda)(2c + \lambda) < 0$

$\lambda > 2|c|$ (5)

25

$\alpha + \beta = \lambda$ (5) $\alpha\beta = c^2,\ c \neq 0$ (5)

$$\left(\sqrt{\alpha} + \sqrt{\beta}\right)^2 = \alpha + \beta + 2\sqrt{\alpha\beta}$$ (5)

$$\left(\sqrt{\alpha} + \sqrt{\beta}\right) = \sqrt{\lambda + 2|c|}$$ (5)

20

$$\left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}\right) + \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) = \left(\sqrt{\alpha} + \sqrt{\beta}\right) - \frac{\left(\sqrt{\alpha}+\sqrt{\beta}\right)}{\sqrt{\alpha\beta}}$$

$$= \frac{\left(\sqrt{\alpha} + \sqrt{\beta}\right)}{\sqrt{\alpha\beta}}\left(\sqrt{\alpha\beta} - 1\right)$$ (5)

$$= \frac{\sqrt{\lambda+2|c|}}{|c|}(|c| - 1)$$ (5)

$$\left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}\right).\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) = \sqrt{\alpha\beta} - \frac{\sqrt{\alpha}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha\beta}}$$ (5)

$$= \sqrt{\alpha\beta} - \frac{\left(\sqrt{\alpha} + \sqrt{\beta}\right)}{\sqrt{\alpha\beta}} + \frac{1}{\sqrt{\alpha\beta}}$$

$$= |c| - \frac{\lambda}{|c|} + \frac{1}{|c|}$$

$$= \frac{c^2-\lambda+1}{|c|}$$ (5)

Required quadratic equation

$$x^2 - \left\{\left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}\right) + \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)\right\}x + \left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}\right)\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) = 0$$ (10)

$$x^2 - \frac{\sqrt{\lambda+2|c|}}{|c|}(|c| - 1)x + \frac{c^2-\lambda+1}{|c|} = 0$$ (5)

$$|c|x^2 - \sqrt{\lambda + 2|c|}(|c| - 1)x + c^2 - \lambda + 1 = 0$$

35

If the graphs of $y = f(x)$ and $y = g(x)$ intersect on y –axis, then $f(0) = g(0)$ (5)

$$c^2 = c^2 - \lambda + 1$$

$$\lambda = 1$$ (5)

10

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11. (b) $F(x) \equiv k(x-1)(x-2)(x-3) + 11x - 6; k \in \mathbb{R}$ (15)

$f(0) = 0$

$k(-1)(-2)(-3) - 6 = 0$ (5)

$k = (-1)$ (5)

$F(x) \equiv -1(x-1)(x-2)(x-3) + 11x - 6$ (5)

$F(x) \equiv -x^3 + 6x^2$ (5) **35**

$F'(x) = -3x^2 + 12x$ (10)

$= -3(x^2 - 4x)$

$= -3[(x-2)^2 - 4]$ (5)

$= 12 - 3(x-2)^2$ (5)

$\therefore [F'(x)]_{maximum} = 12$ (5) **25**

12. (a) Let P_1, P_2 and P_3 be three sets given by $P_1 = \{\propto, \beta, \gamma\}$, $P_2 = \{A, B, C\}$, $P_3 = \{1,3,5\}$ respectively. It is required to form a **five**-character password using elements belongs to $P_1 \cup P_2 \cup P_3$.

(i) Find the number of different element passwords that can be formed using above different elements

I. when there are no restrictions on the selection

II. consisting exactly two digits

(ii) Find the number of different passwords that can be formed using above elements such that exactly one element from P_1 and exactly two elements from P_2 with repetitions allowed only for elements from P_3

(b) Let $U_r = \frac{9(3r+4)}{(3r-1)(3r+2)(3r+5)}$ and $f(r) = \frac{A}{(3r-1)} + \frac{B}{(3r+2)}$ for $r \epsilon \mathbb{Z}^+$, where A and B are real constants.

Let $U_r = f(r) - f(r+1)$ for $r \epsilon \mathbb{Z}^+$, show that $A = \frac{5}{2}$ and $B = \frac{1}{2}$.

Hence, show that $\sum_{r=1}^{n} U_r = \frac{27}{20} - \frac{5}{2(3n+2)} - \frac{1}{2(3n+5)}$; $n \epsilon \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Hence, find the greatest real constant k such that $\sum_{r=1}^{\infty} (U_r + kU_{r+2}) \leq 1$

12. (a) (i) $P_1 = \{\alpha, \beta, \gamma\}$ $P_2 = \{A, B, C\}$ $P_3 = \{1,3,5\}$

I. *when there are no restriction on the selection*

$= {}^{9}C_5 \times 5! = \frac{9!}{4! \times 5!} = 15{,}120$ (5) (5)

10

II. *when selecting 2 digit exactly;*

$= {}^{6}C_3 \times 3C_2 \times 5!$

$= \frac{6!}{3! \times 3!} \times 3! \times 5! = \frac{6 \times 5 \times 4 \times 120 \times 3}{3 \times 2 \times 2} = 7200$ (5) (5) (5)

Alter method

P_1	P_2	P_3
1	2	2
2	1	2
3	0	2
0	3	2

$3C_1 \times {}^{3}C_2 \times 3C_1 \times 5!$

$3C_2 \times {}^{3}C_1 \times 3C_2 \times 5!$

$3C_3 \times {}^{3}C_0 \times 3C_2 \times 5!$

$3C_0 \times {}^{3}C_3 \times 3C_2 \times 5!$

Total diffrent Passward $= 2 \times 3 \times 3 \times 3 \times 5! + 2 \times 1 \times 1 \times 3 \times 5! = 7200$

15

(ii)

	P_1	P_2	P_3
(10)	1	2	*two different*
(10)	1	2	*two equal*

(05) $\Rightarrow 3C_1 \times {}^{3}C_2 \times 3C_2 \times 5!$

$\Rightarrow 3C_1 \times {}^{3}C_2 \times 3C_1 \times \frac{5!}{2!}$ (05)

$\therefore$ *total number of different passwards* $= 3C_1 \times {}^{3}C_2 \times \left[{}^{3}C_2 + \frac{3C_1}{2}\right] \times 5!$ (05)

$= 3 \times 3\left[3 + \frac{3}{2}\right] \times 5 \times 4 \times 3$

$= 9 \times \frac{9}{2} \times 60$ (05)

$= 81 \times 30 = 2430$ (05)

45

(b) $U_r = f(r) - f(r+1)$

$\frac{9(3r+4)}{(3r-1)(3r+2)(3r+5)} = \frac{A}{(3r-1)} + \frac{B}{(3r+2)} - \frac{A}{(3r+2)} - \frac{B}{(3r+5)}$ (05)

$9(3r+4) \equiv A(3r+2)(3r+5) + (B-A)(3r-1)(3r+5) - B(3r-1)(3r+2)$

$\equiv (9A+9B)r + 15A - 3B$ (05)

$r; 27 = 9A + 9B \longrightarrow (1)$ (05)

$3 = A + B$

$r^0; 36 = 15A - 3B \longrightarrow (2)$ (05)

(1), (2)

$45 = 18A$

$A = \frac{5}{2}$

$B = \frac{1}{2}$

20

$\therefore f(r) = \frac{5}{2(3r-1)} + \frac{1}{2(3r+2)}$ (05)

$u_r = f(r) - f(r+1)$

$r = 1;$ $u_1 = f(1) - f(2)$

$r = 2;$ $u_2 = f(2) - f(3)$ (05)

$r = 3;$ $u_3 = f(3) - f(4)$

… … … … … … … ….

… … … … … … … … ..

… . . … … … … … … …

$r = n-2;$ $u_{n-2} = f(n-2) - f(n-1)$

$r = n-1;$ $u_{n-2} = f(n-1) - f(n)$ (05)

$r = n;$ $u_n = f(n) - f(n+1)$

$\sum_{r=1}^{n} u_r = f(1) - f(n+1)$ (05)

$= \frac{5}{4} + \frac{1}{10} - \frac{5}{2[3(n+1)-1]} - \frac{1}{2[3(n+1)+2]}$ (05)

$= \frac{27}{20} - \frac{5}{2(3n+2)} - \frac{1}{2(3n+5)}$ (05)

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(05) $$\lim_{n\to\infty}\sum_{r=1}^{n} U_r = \lim_{n\to\infty}\left[\frac{27}{20} - \frac{5}{2(3n+2)} - \frac{1}{2(3n+5)}\right]$$

$$= \frac{27}{20}$$ (05)

$\therefore$ *the infinite series* $\sum_{r=1}^{\infty} U_r$ *is convergent and the sum* $= 27/20$ (05)

$\sum_{r=1}^{\infty} U_r$

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$$\sum_{r=1}^{\infty}(U_r + kU_{r+2}) \le 1$$

$$\sum_{r=1}^{\infty} U_r + k\{\underbrace{[U_3 + U_4 + U_5 + \cdots] + U_1 + U_2}\} - kU_1 - kU_2 \le 1$$

From u_r,

$$U_1 = \frac{9\times 7}{2\times 5\times 8}$$

$$U_2 = \frac{9\times 10}{5\times 8\times 11}$$

$$(1+k)\sum_{r=1}^{\infty} U_r - k(U_1 + U_2) \le 1$$ (05)

$$(1+k)\frac{27}{20} - k\left(\frac{9}{5\times 8}\left(\frac{7}{2} + \frac{10}{11}\right)\right) \le 1$$

$$(1+k)\frac{27}{20} - k\left(\frac{9}{40}\times\frac{97}{22}\right) \le 1$$

$$\frac{27}{20} + \left(\frac{27}{20} - \frac{9}{40}\times\frac{97}{22}\right)k \le 1$$

$$\frac{27}{20} + \frac{9k}{20}\left(3 - \frac{97}{44}\right) \le 1$$ (05)

$$27 + \frac{9k}{44}\times 35 \le 20$$

$$9k \times \frac{35}{44} \le -7$$

$$k \le -\frac{44}{45}$$

$\therefore$ *Greatest value of* $k = -\frac{44}{45}$ (05)

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13. (*a*) (i) Let $(\mathbf{PQ})^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}$ for two matrices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{A}, \mathbf{B}$ be two square matrices of same order. If $\mathbf{A}$ is a symmetric matrix, prove that $\mathbf{B}^{\mathrm{T}}\mathbf{AB}$ is symmetric and if $\mathbf{A}$ is skew symmetric $\mathbf{B}^{\mathrm{T}}\mathbf{AB}$ is skew symmetric.

(ii) Let the matrix $\mathbf{A}(x) = (1 - x^2)^{-\frac{1}{2}}\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$ for $x \in \mathbb{R}$, where $|x| < 1$.

Given that $\mathbf{A}(x)\mathbf{A}(y) = \mathbf{A}(z)$, determine z in terms of x and y. Using a suitable substitution for z find the inverse matrix of $\mathbf{A}(x)$.

(*b*) Point P represents a complex number z, where $z = r(\cos\theta + i\sin\theta); r > 0, -\pi < \theta \leq \pi$ in an Argand diagram. Provide geometric constructions for the point Q which represent z^2.

Draw the locus described by $|1 + z| = 1$, **hence** find z such that $|z + 1| = 1$ and $|z^2 + 1| = 1$.

(*c*) Let $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Using **De Moivre's theorem**, find ω^5.

Prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

Hence, find the value of $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}$.

13(*a*) (i). If A is symmetric $A^T = A$ (05)

$$(B^TAB)^T = B^TA^T(B^T)^T$$

$$= B^TAB \quad (05)$$

$\therefore B^TAB$ is a symmetric matrix

If A is skew symmetric $A^T = -A$ (05)

$$(B^TAB)^T = B^TA^T(B^T)^T$$

$$= B^T(-A)B$$

$$= -B^TAB \quad (05)$$

$\therefore B^TAB$ is a skew symmetric matrix

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(ii.) $|x| < 1 \Longrightarrow (1-x^2)^{-\frac{1}{2}} \in \mathbb{R}^+$ (05)

$$A(x)A(y) = (1-x^2)^{-\frac{1}{2}}\begin{pmatrix}1 & x\\ x & 1\end{pmatrix}(1-y^2)^{-\frac{1}{2}}\begin{pmatrix}1 & y\\ y & 1\end{pmatrix}$$

$$= (1-x^2-y^2+x^2y^2)^{-\frac{1}{2}}\begin{pmatrix}1 & x\\ x & 1\end{pmatrix}\begin{pmatrix}1 & y\\ y & 1\end{pmatrix}$$ (05)

$$= (1+xy)(1-x^2-y^2+x^2y^2)^{-\frac{1}{2}}\begin{pmatrix}1 & \frac{x+y}{1+xy}\\ \frac{x+y}{1+xy} & 1\end{pmatrix}$$ (05)

$$= \left(\frac{(1+xy)^2-(x+y)^2}{(1+xy)^2}\right)^{-\frac{1}{2}}\begin{pmatrix}1 & \frac{x+y}{1+xy}\\ \frac{x+y}{1+xy} & 1\end{pmatrix}$$

$$= \left(1-\left(\frac{x+y}{1+xy}\right)^2\right)^{-\frac{1}{2}}\begin{pmatrix}1 & \frac{x+y}{1+xy}\\ \frac{x+y}{1+xy} & 1\end{pmatrix}$$ (05)

$$A(x)A(y) = (1-z^2)^{-1/2}\begin{pmatrix}1 & z\\ z & 1\end{pmatrix}$$ (05)

$$= A(z)$$

Where $z = \frac{x+y}{1+xy}$ (05)

$$1-z^2 = 1-\left(\frac{x+y}{1+xy}\right)^2 = \frac{(1-x^2)}{(1+xy)^2} > 0$$

$\therefore |z| < 1$

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Let $z = 0 \Longrightarrow x+y=0 \Rightarrow y=-x$ (05)

$$A(x)A(-x) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$ (05)

$$A^{-1}(x) = A(-x)$$

$$= (1-x^2)^{-\frac{1}{2}}\begin{pmatrix}1 & -x\\ x & 1\end{pmatrix}$$ (05)

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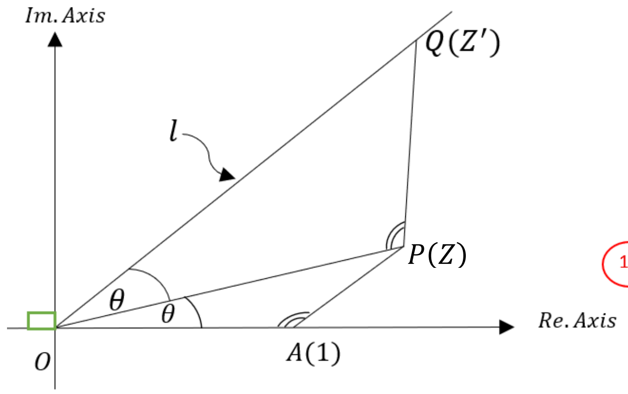
$(\boldsymbol{b})$

Let $z = r(\cos\theta + i\sin\theta)$

$z^2 = r^2(\cos\theta + i\sin\theta)^2$

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$ (05)

Point Q is marked on the line l such that $O\hat{A}P = O\hat{P}Q$



(10)

OAP and OPQ are similar triangles

$\frac{OQ}{OP} = \frac{OP}{OA}$

$OQ = OP^2 = r^2$ (05)

$Arg(z') = 2\theta$, $|z' = r^2|$

$\therefore z' = z^2$

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Let $z = r(\cos\theta + i\sin\theta), OP = r$

$OA = AP = AQ = 1, OQ = r^2$ (05)

$OP = 2OA\cos(\pi - \theta) = -2\cos\theta$

$r = -2\cos\theta \longrightarrow (1)$ (05)

$OQ = 2OA\cos(2\theta - \pi) = 2\cos(\pi - 2\theta)$

$r^2 = -2\cos 2\theta \longrightarrow (2)$ (05)

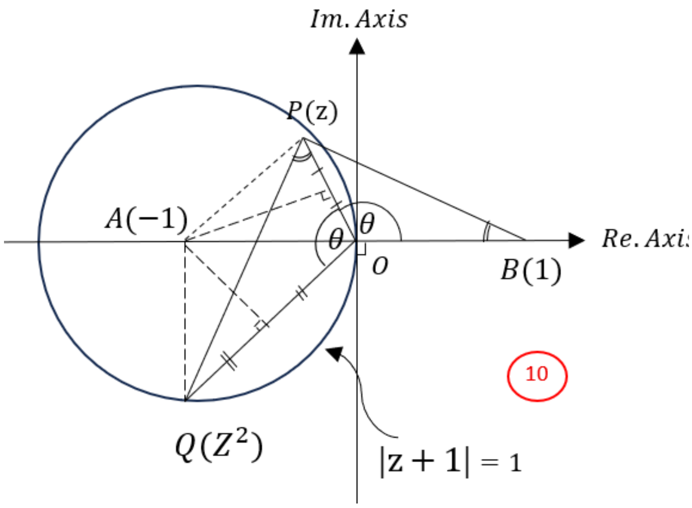
$(1),(2) \Rightarrow \ 4\cos^2\theta = -2\cos 2\theta$

$1 + \cos 2\theta = -\cos 2\theta$

$\cos 2\theta = -\frac{1}{2} \Longrightarrow 2\theta = \pm 240^0$ (05)

$\theta = \pm 120^0 \Longrightarrow \quad r = -2\cos(\pm 120^0) = 1$ (05)

$z = 1[\cos(\pm 120^0) + i\sin(\pm 120^0)] = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ (05)



(10)

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13(c) $\Rightarrow x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$

$\omega^5 - 1 = (\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1)$

$\omega \neq 1$ (05)

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

(05)

$\omega^2 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = \cos\left(\pi - \frac{\pi}{5}\right) + i\sin\left(\pi - \frac{\pi}{5}\right)$

$= -\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$\omega^3 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \cos\left(\pi + \frac{\pi}{5}\right) + i\sin\left(\pi + \frac{\pi}{5}\right) = -\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$ (05)

$\omega^4 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right) = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$ (05)

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} - \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} - \cos\frac{\pi}{5} - i\sin\frac{\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = 0$

$\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$ (05)

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14. (*a*) Let $f(x) = \frac{x(x-4)}{(x+4)^2}$ for $x \epsilon \mathbb{R} - \{-4\}$

Show that $f'(x) = \frac{4(3x-4)}{(x+4)^3}$, for $x \neq -4$, where $f'(x)$ is the first derivative of $f(x)$. **Hence**, find the intervals on which $f\,(x)$ is increasing and the interval on which $f\,(x)$ is decreasing. Sketch the graph of $y = f\,(x)$ indicating asymptotes, turning points, and points of intersection on the x, y axes.

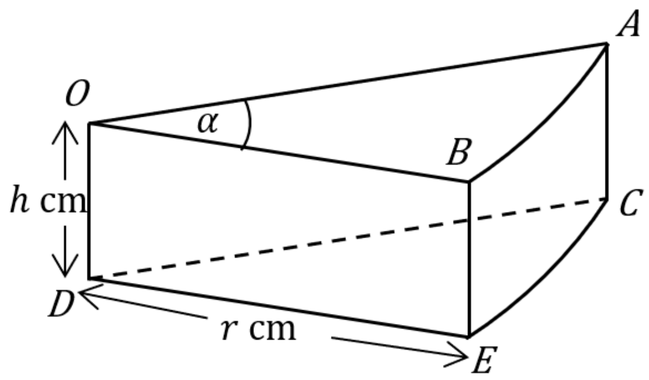
It is given that $f''(x) = \frac{-24(x-4)}{(x+4)^4}$; $x \neq -4$.

Show that $(4,0)$ is the point of inflection of $y = f(x)$.

Deduce the graph of

$$y = \frac{x(x+4)}{(x+8)^2} \text{ ; for } x < -8$$

(*b*) The adjoining figure shows a cheese wedge consisting two identical plane faces OAB and DCE which are sectors of a circle of radius r cm with sector angle α radians, where $0 < \alpha < \frac{\pi}{2}$. The three edges OD, BE, and AC are all perpendicular to both faces giving the solid appearance as shown in figure with thickness of h cm. Given that the volume of the cheese wedge is 9 cm^3, show that the surface area of the wedge is given by $S = r^2\alpha + \frac{18(2+\alpha)}{r\alpha}$. Find $\frac{ds}{dr}$. **Hence,** find r such that the area S is minimum for $\alpha = 1$ radians.



14(a) for $x \neq -4$, $f(x) = \frac{x(x-4)}{(x+4)^2}$ (05)

$$f'(x) = \frac{(x+4)^2\frac{d(x^2-4x)}{dx} - x(x-4)\frac{d(x+4)^2}{dx}}{(x+4)^4}$$ (05)

$$f'(x) = \frac{(x+4)^2(2x-4)-x(x-4).2(x+4)}{(x+4)^4}$$

$$f'(x) = \frac{2(x+4)[(x+4)(x-2)-x^2+4x]}{(x+4)^4}$$

$$f'(x) = \frac{2(x^2+2x-8-x^2+4x)}{(x+4)^3}$$ WLA (05)

$$f'(x) = \frac{4(3x-4)}{(x+4)^3}$$

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When $\lim_{x\to-\infty} f(x) = 1$, when $\lim_{x\to+\infty} f(x) = 1$ } $y = 1$ *is a horizontal asymptote* (05)

when $\lim_{x\to-4^-} f(x) = +\infty$, *when* $\lim_{x\to-4^+} f(x) = +\infty$ } $\therefore x = -4$ *is a vertical asymptote* (05)

when $x = 0, y = 0$ (05)

when $y = 0, x = 0$ *or* $x = 4$ (05) *when* $f'(x) = 0 \Rightarrow x = 4/3$ (05)

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x	$x < -4$	$-4 < x < \frac{4}{3}$	$x > 4$
sign of $f'(x)$	$+$	$-$	$+$
function	*Increasing*	*decreasing*	*Increasing*

(15)

when $x = \frac{4}{3}$, $f(x)$ *has a relative Minimum* $y = -\frac{1}{8}$

Minimum point $(\frac{4}{3}, \frac{-1}{8})$ (05)

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when $f''(x) = 0$, there may be a point of inflection

$x = 4, y = 0$

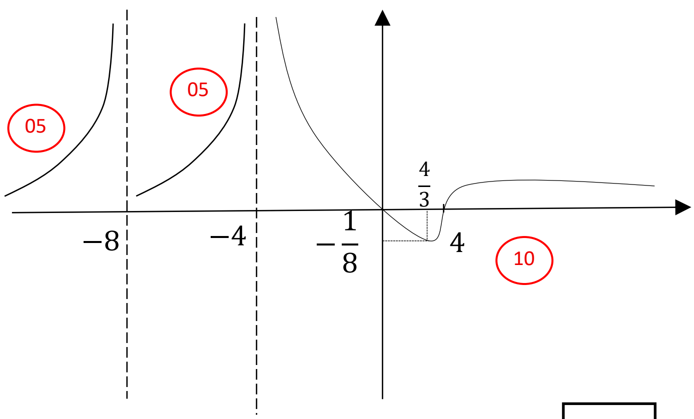
x	$x < -4$ (05)	$-4 < x < 4$ (05)	$x > 4$ (05)
sign of $f''(x)$	$-$	$-$	$+$
concavity	down	down	up

Concavity changes at $x = 4$.
$\therefore$ (4,0) is a point of inflection. (05)

$y = \frac{x(x+4)}{(x+8)} = \frac{(x+4-4)(x+4)}{(x+4+4)}$

$X = x + 4$

(05)



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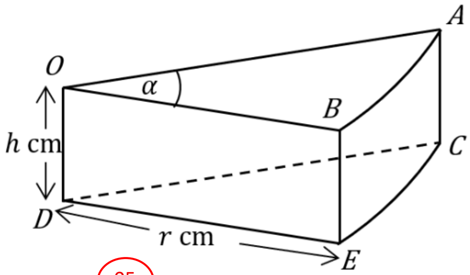
(**b**)



r	$0 < r < 3$ (05)	$r > 3$ (05)
$\frac{ds}{dr}$	$-$	$+$

$r = 3$

(05)

$$S = 2hr + \frac{1}{2}r^2\alpha \times 2 + 2\pi r \times \frac{\alpha}{2\pi} \times h$$ (05)

$$\therefore S = 2hr + r^2\alpha + \alpha rh$$

But the volume of the (05)

cheese wedge $9 = \pi r^2 \times \frac{\alpha}{2\pi}h \Longrightarrow h = \frac{18}{\alpha r^2}$

(05)

$$\therefore \quad S = 2r \times \frac{18}{\alpha r^2} + r^2\alpha + \alpha r.\frac{18}{\alpha r^2} = \frac{36}{\alpha r} + r^2\alpha + \frac{18}{r} = \frac{18}{r\alpha}(\alpha + 2) + r^2\alpha$$

$\frac{d(S)}{dr} = \frac{18(\alpha+2)}{\alpha}.(-1)\frac{1}{r^2} + 2r\alpha$ (05)

$\frac{d(s)}{dr} = 0, \alpha = 1$ (05) $\quad \frac{3\times18}{1}.\frac{1}{r^2} = 2 \times 1 \times r \Longrightarrow r^3 = 27, \therefore r = 3cm$ (05)

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15. (a) Find the values of the constants A, B and C such that $x^2 + 2 \equiv A(x^2 - 2x + 4) + (Bx + C)(x + 2)$ for all $x \epsilon \mathbb{R}$,

Hence, integrate $\int \frac{x^2 + 2}{x^3 + 8}\,dx$

(b) By using a suitable substitution prove that $\int_a^b f(x)dx = \int_a^b f(a + b - x)\,dx$, where a and b are constants.

Let $I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\text{Sin } x}{\text{Cos } x - \text{Sin } x}\,dx$. Show that $I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{-\text{Sin } x}{\text{Cos } x + \text{Sin } x}\,dx$.

Deduce that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \tan 2x \; dx = 0$

Find the values of constants λ and μ such that $\text{Sin x} = \lambda(\cos x - \sin x) - \mu(\cos x + \sin x)$,

show that $I = \ln(\frac{\sqrt{3}-1}{\sqrt{2}}) - \frac{\pi}{6}$.

(c) Let $y = \sec^2 x - \tan x$. Find $\frac{dy}{dx}$.

Hence, evaluate. $\int_0^{\frac{\pi}{3}} \sec^2 x\,(\tan x - 1)\; dx$

(a) $x^2+2 = A(x^2-2x+4)+(Bx+c)(x+2)$

$x^2 \Longrightarrow A+B=1 \longrightarrow 1)$

$x^1 \Longrightarrow -2A+2B+C=0 \longrightarrow (2)$ $\quad A=B=\frac{1}{2}$ (5) $\quad C=0$ (5)

$x^0 \Longrightarrow 4A+2C=2 \longrightarrow (3)$

$\therefore \int \frac{x^2+2}{x^3+8}dx = \int \frac{A(x^2-2x+4)+Bx(x+2)}{x^3+8}dx$

$= A\int \frac{(x^2-2x+4)}{(x+2)(x^2-2x+4)}dx + B\int \frac{x(x+2)}{(x+2)(x^2-2x+4)}dx$

$= A\int \frac{1}{x+2}dx + B\int \frac{x}{x^2-2x+4}dx$ (5) (5)

$= A\ln|x+2| + B\frac{1}{2}\int \frac{2x-2+2}{x^2-2x+4}dx$ (5)

$= A\ln|x+2| + \frac{B}{2}\int \frac{2x-2}{x^2-2x+4}dx + B\int \frac{1}{x^2-2x+4}dx$ (5) (5) (5)

$= A\ln|x+2| + \frac{B}{2}\ln|x^2-2x+4| + B\int \frac{1}{\sqrt{3}^2+(x-1)^2}dx$

$= \frac{1}{2}\ln|x+2| + \frac{1}{4}\ln|x^2-2x+4| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+\lambda$ (5) (5)

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(b) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ ***proof***

Let, $x = a+b-X$

$\therefore dx = -dX$

$\begin{pmatrix} x=a \\ X=b \end{pmatrix} \quad \begin{pmatrix} x=b \\ X=a \end{pmatrix}$ (5)

$\int_a^b f(x)dx = \int_b^a f(a+b-X)(-dX)$

$= -\int_b^a f(a+b-X)dX$ (5)

$= \int_a^b f(a+b-X)dX$

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(5)

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$I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\cos x - \sin x}dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\left(\frac{\pi}{3}+\frac{2\pi}{3}-x\right)}{\cos\left(\frac{\pi}{3}+\frac{2\pi}{3}-x\right)-\sin\left(\frac{\pi}{3}+\frac{2\pi}{3}-x\right)}dx$ (5)

$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{-\cos x - \sin x}dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{-\sin x}{\cos x + \sin x}dx$ (5)

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\cos x - \sin x}\,dx + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\cos x + \sin x}\,dx = 0$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x(\cos x + \sin x) + \sin x(\cos x - \sin x)}{(\cos x - \sin x)(\cos x + \sin x)}\,dx = 0$$

(5)

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x}\,dx = 0 \Longrightarrow \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \tan 2x\,dx = 0$$

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$$\sin x = \lambda(\cos x - \sin x) - \mu(\cos x + \sin x)$$

$\sin x \Longrightarrow -\lambda - \mu = 1$

$\cos x \Longrightarrow \lambda - \mu = 0$ (5) $\lambda = \mu = -\frac{1}{2}$ (5)

$$I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\cos x - \sin x}\,dx$$

$$I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\lambda(\cos x - \sin x) - \mu(\cos x + \sin x)}{\cos x - \sin x}\,dx$$

$$I = \lambda \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} dx + \mu \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{-\cos x - \sin x}{\cos x - \sin x}\,dx$$

(5) (5)

$$= \left[\lambda x + \mu \ln|\cos x - \sin x|\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$ (5)

$$= -\frac{1}{2}\left[\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) + \ln\left|\frac{\cos\frac{2\pi}{3} - \sin\frac{2\pi}{3}}{\cos\frac{\pi}{3} - \sin\frac{\pi}{3}}\right|\right]$$ (5)

$$= -\frac{1}{2}\left[\frac{\pi}{3} + \ln\left|\frac{-\frac{1}{2} - \frac{\sqrt{3}}{2}}{\frac{1}{2} - \frac{\sqrt{3}}{2}}\right|\right]$$

$$= -\frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{\pi}{6}$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) - \frac{\pi}{6}$$ (5)

$$= \frac{1}{2}\ln\left(\frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}\right) - \frac{\pi}{6}$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 - \frac{\pi}{6} = \ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) - \frac{\pi}{6}$$

35

(**C**) $y = \sec^2 x - \tan x$

$$\frac{dy}{dx} = 2\sec x \sec x \tan x - \sec^2 x$$

$$= \sec^2 x\,(2\tan x - 1)$$ (5)

$$I = \int_0^{\frac{\pi}{3}} \sec^2 x(\tan x - 1)dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec^2 x(2\tan x - 2)dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec^2 x(2\tan x - 1 - 1)dx$$

5 5

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec^2 x(2\tan x - 1)dx - \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec^2 x dx$

5

$= \frac{1}{2}[y]_0^{\frac{2\pi}{3}} - \frac{1}{2}[\tan x]_0^{\frac{2\pi}{3}}$ 5

$= \frac{1}{2}[\sec^2 x - \tan x]_0^{\frac{\pi}{3}} - \frac{1}{2}\left(\tan\frac{\pi}{3} - \tan 0\right)$ 5

$= \frac{1}{2}[((2)^2 - \sqrt{3}) - ((1)^2 - 0)] - \frac{\sqrt{3}}{2}$

$= \frac{1}{2}[3 - \sqrt{3} - \sqrt{3}]$

$= \frac{\sqrt{3}}{2}(\sqrt{3} - 2)$ 5

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16. Let $A \equiv (1,2)$ is on the line $l \equiv ax + by - 3 = 0$ where $a, b \in \mathbb{R}$ and the parametric equation of the line l is given by $x = 2t + 1, y = t + 2$ where t is a parameter. Find the values of a and b.

Let the circle $S \equiv x^2 + y^2 - 4x - 5y + 9 = 0$ cuts the line l at A and B. Find the coordinates of the point B.

Show that the centre of the circle S lies on l and find the equations of tangents drawn from A to the circle S_1 with centre B and radius $\frac{\sqrt{10}}{2}$.

Find the angle between one of the above tangents and line l and find the area bounded by the above two tangents and the common chord of the circles S and S_1.

$l; ax + by - 3 = 0$

Passing through $A \equiv (1,2)$

$a + 2b + 3 = 0$ 5

$a + 2b = -3 \longrightarrow (1)$

Parametric form $2t + 1, t + 2$

Gradient $\frac{t+2-2}{2t+1-1} = -\frac{a}{b}$ 5

$\frac{t}{2t} = -\frac{a}{b}$

$2a + b = 0 \longrightarrow (2)$ 5

$(1) \times 2 - (2) \Longrightarrow 3b = 6 \Longrightarrow b = 2$

$a = +1$ 5

$\therefore l \equiv x - 2y + 3 = 0$

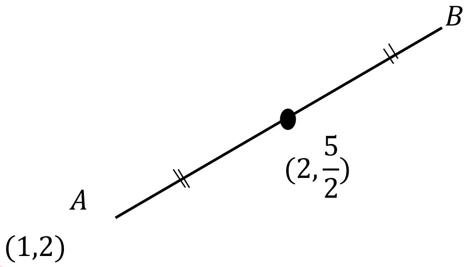
20

$S = x^2 + y^2 - 4x - 5y + 9 = 0$

Centre $\left(2, \frac{5}{2}\right)$ (5)

$x - 2y = 2 - 2 \times \frac{5}{2} + 3 = 0$ (5)

$\therefore$ *centre on* $l = 0$



Coordinates of $B = \frac{2\times2-1\times1}{2-1}, \frac{2\times\frac{5}{2}-1\times2}{2-1} = (3,3)$ (5) (5)

20

---

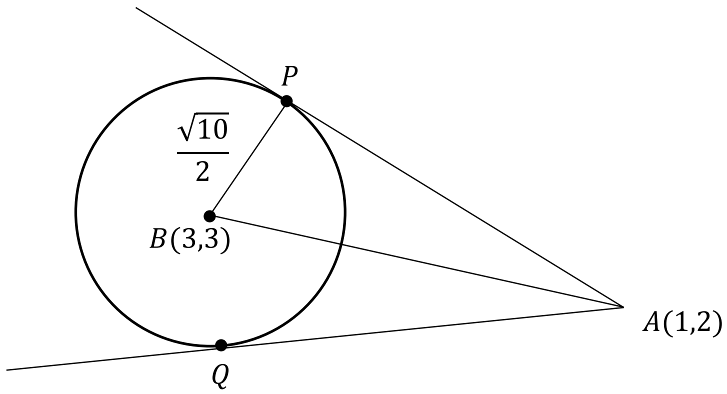
let $m_{AP} = m$

$\therefore AP; \frac{y-2}{x-1} = m$ (5)

$y - 2 = mx - m$

$mx - y + (2 - m) = 0$ (5)

$\perp$ *distance form B to AP* (5)



$\frac{|m(3)-3+(2-m)|}{\sqrt{m^2+1}} = \frac{\sqrt{10}}{2}$

$\frac{|3m-3+2-m|}{\sqrt{m^2+1}} = \frac{\sqrt{10}}{2}$

(5) $\frac{|2m-1|}{\sqrt{m^2+1}} = \frac{\sqrt{10}}{2}$ (5)

$4(2m-1)^2 = 10(m^2+1)$

$16m^2 - 16m + 4 = 10m^2 + 10$

$6m^2 - 16m - 6 = 0$

$3m^2 - 8m - 3 = 0$ (5)

$(3m+1)(m-3) = 0$

$m = -\frac{1}{3}$ (5) *or* $m = 3$ (5)

$\therefore$ *Equation of tangents are*

$m = -\frac{1}{3} \Longrightarrow -\frac{1}{3}x - y + 2 + \frac{1}{3} = 0$

$x + 3y - 7 = 0$ (5)

$m = 3 \Rightarrow 3x - y + (2 - 3) = 0$

$3x - y - 1 = 0$ (5)

50

Angle between, $l = 0$ *and* $3x - y - 1 = 0$

let the angle is θ

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ (5)

$= \left|\frac{\frac{1}{2} - 3}{1 + \frac{1}{2}(3)}\right| = 1$ (5) (5)

$\theta = 45^0$ (5)

these two tangents are $\perp$ *to each othes*

$\therefore$ *Angle between,* $l = 0$ *and one of the tangent is* 45^0 (5)

25

$S_1 = (x-3)^2 + (y-3)^2 = \left(\frac{\sqrt{10}}{2}\right)^2$ (5)

$x^2 + y^2 - 6x - 6y + 18 - \frac{10}{4} = 0$

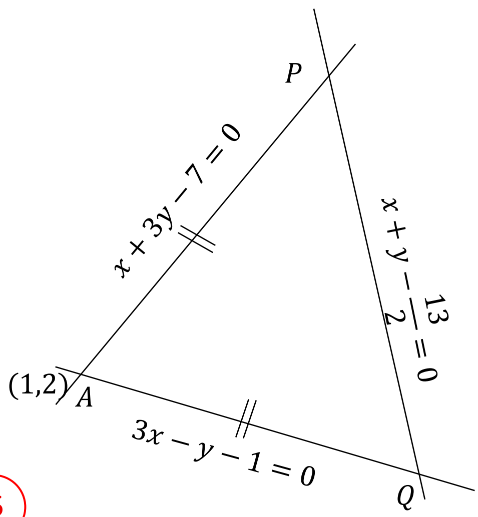
$S_1 = x^2 + y^2 - 6x - 6y + \frac{31}{2} = 0$

$S = x^2 + y^2 - 4x - 5y + 9 = 0$

equation of common chord

$S - S_1 = 0$

$2x + y - \frac{13}{2} = 0$ (5)



$P; x + y - \frac{13}{2} = 0 \longrightarrow 1$

$x + 3y - 7 = 0 \longrightarrow 2$

$2 - 1 \Rightarrow 2y - \frac{1}{2} = 0$

$y = \frac{1}{4}$ (5) $\therefore x = \frac{25}{4}$ (5)

$P = (\frac{25}{4}, \frac{1}{4})$

$AP^2 = \left(\frac{25}{4} - 1\right)^2 + \left(\frac{1}{4} - 2\right)^2$ (5)

$= \left(\frac{21}{4}\right)^2 + \left(-\frac{7}{4}\right)^2$

$= \frac{441+49}{16} = \frac{490}{16}$ (5)

$AP = \frac{7}{4}\sqrt{10}$

$\therefore$ *Area* $= \frac{1}{2} \times \left(\frac{7}{4}\sqrt{10}\right)^2$

$= \frac{49\times10}{2\times16} = \frac{245}{16}$ *sq. units* (5)

35

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17. (a) Prove that $\cot\theta \equiv \operatorname{cosec} 2\theta + \cot 2\theta$ for $\theta \neq \frac{n\pi}{2}$ and $n \in \mathbb{Z}$. **Hence,** show that $\cot\frac{\pi}{12} = 2 + \sqrt{3}$.

Hence, find $\operatorname{cosec}^2\frac{\pi}{12}$. Determine x and y positive integers such that $\operatorname{cosec}\frac{\pi}{12} = \sqrt{x} + \sqrt{y}$.

Deduce the value of $tan\frac{\pi}{24}$

(b) In the usual notation, state and prove the **Sine rule** for any triangle.

In a triangle ABC, it is given that $BC = 5$ cm, $AC = 4$ cm, and $A = \frac{\pi}{2} + B$. Show that $\tan C = \frac{9}{40}$.

Hence, find $\sin B$ and length AB. The internal bisector of the angle $B\hat{A}C$ meets the side BC at D. Find length BD.

(c) Solve $\cos^{-1}(1 - x) = 2\sin^{-1}x$.

$$\operatorname{cosec} 2\theta + \cot 2\theta \equiv \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta} \equiv \frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta} \equiv \cot\theta$$ (05) (05)

$$\cot\theta = \operatorname{cosec} 2\theta + \cot 2\theta$$

$when\ \theta = \frac{\pi}{12},$ $\quad \cot\frac{\pi}{12} = \operatorname{cosec}\frac{\pi}{6} + \cot\frac{\pi}{6} = 2 + \sqrt{3}$ (05)

$$\operatorname{cosec}^2\frac{\pi}{12} = 1 + \cot^2\frac{\pi}{12} = 1 + (2 + \sqrt{3})^2$$ (05) (05)

$$\operatorname{cosec}\frac{\pi}{12} = \sqrt{8 + 4\sqrt{3}}$$ (05)

$$\sqrt{8 + 4\sqrt{3}} = \sqrt{x} + \sqrt{y}$$

$$8 + 2\sqrt{12} = x + y + 2\sqrt{xy}$$

$\therefore x + y = 8 \longrightarrow (1)$ (05)

$xy = 12 \longrightarrow (2)$ (05)

$By\ solving\ (1)\ and\ (2)$

$x = 2 \rightarrow y = 6$

$x = 6 \rightarrow y = 2$ (05)

$\therefore \sqrt{8 + 4\sqrt{3}} = \sqrt{2} + \sqrt{6}$

$\operatorname{cosec}\frac{\pi}{12} = \sqrt{2} + \sqrt{6}$

$$\cot\frac{\pi}{12} = 2 + \sqrt{3}$$

$$\therefore \tan\frac{\pi}{12} = \frac{1}{2 + \sqrt{3}}$$

$$\tan\frac{\pi}{12} = 2 - \sqrt{3}$$

$\cot\theta = \operatorname{cosec} 2\theta + \cot 2\theta$

(05)

$$\tan\theta = \frac{1}{(\operatorname{cosec} 2\theta + \cot 2\theta)} \cdot \frac{(\operatorname{cosec} 2\theta - \cot 2\theta)}{(\operatorname{cosec} 2\theta - \cot 2\theta)}$$

$$\tan\theta = \frac{(\operatorname{cosec} 2\theta - \cot 2\theta)}{(\operatorname{cosec}^2 2\theta + \cot^2 2\theta)}$$

$\tan\theta = (\operatorname{cosec} 2\theta - \cot 2\theta)$ (05)

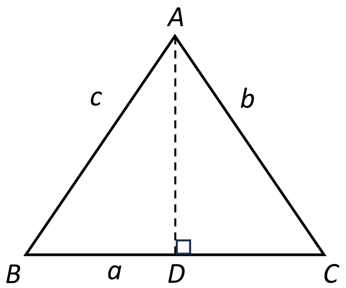
when $\theta = \frac{\pi}{24}, \tan\frac{\pi}{24} = \operatorname{cosec}\frac{\pi}{12} - \cot\frac{\pi}{12}$

$\tan\frac{\pi}{24} = \sqrt{2} + \sqrt{6} - (2 + \sqrt{3})$

$\tan\frac{\pi}{24} = \sqrt{2} - \sqrt{3} + \sqrt{6} - \sqrt{2}$ (05)

60

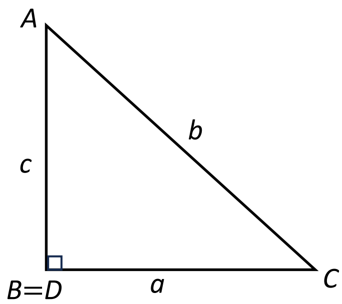
(b)



(i) ABC is an acute angled triangled.

$AD = AB\sin B = AC\sin C$

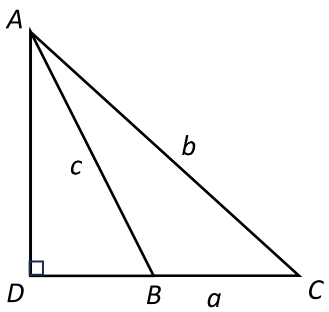
$\because \frac{b}{\sin B} = \frac{c}{\sin C}$ (05)



(ii) ABC is a right angled triangled.

$AD = AB\sin B = AC\sin C$

$\because \frac{b}{\sin B} = \frac{c}{\sin C}$ (05)



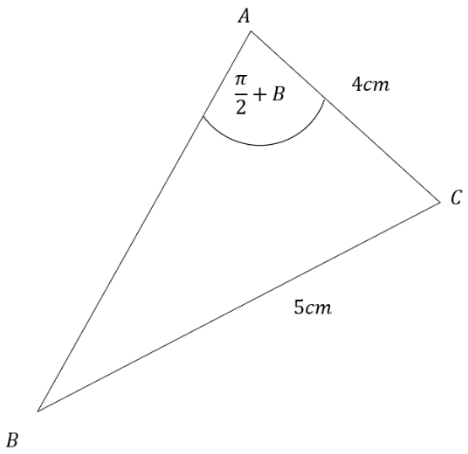
(iii) ABC is an obtuse angled triangled.

$AD = AB\sin(\pi - B) = AC\sin C$

$\because \frac{b}{\sin B} = \frac{c}{\sin C}$ (05)

Similarly, we can show $\frac{a}{\sin A} = \frac{c}{\sin C}$ (05)

$$\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$$\frac{4}{\sin B}=\frac{5}{\sin\left(\frac{\pi}{2}+B\right)}$$ 05

$\frac{4}{5}=tanB$ 05

$A+B+C \quad =\pi$

$\frac{\pi}{2}+B+B+C=\pi$

$C=\frac{\pi}{2}-2B$ 05

$tanC=tan\left(\frac{\pi}{2}-2B\right)$

$=cot2B$ 05

$=\frac{1-\tan^2 B}{2tanB}=\frac{1-16/25}{8/5}$

$tanC=9/40$

$tanB=\frac{4}{5} \implies \sin B=\frac{4}{\sqrt{41}}$ 05

$Let\ AB=x$

$\frac{x}{sin\ C}=\frac{4}{sin\ B}$ 10

$\frac{x}{sin\left(\frac{\pi}{2}-2B\right)}=\frac{4}{sin\ B}$

$\frac{x}{cos\ 2B}=\frac{4}{sin\ B}$

$\frac{x}{\frac{9}{41}}=\frac{4}{sin\ B}$

$\frac{41x}{9}=\frac{4\sqrt{41}}{4}$

$x=\frac{9}{\sqrt{41}}\ cm$

$\therefore\ AB=\frac{9}{\sqrt{41}}\ cm$ 05

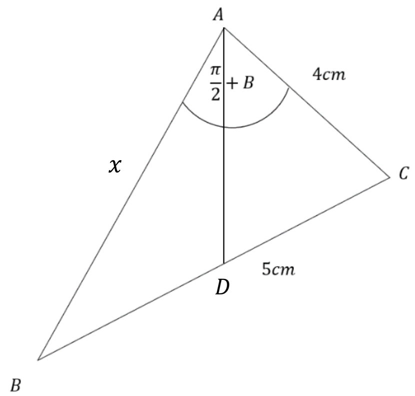
Using angle bisector theorem

$\frac{AB}{AC}=\frac{BD}{DC}$ 05

$\frac{9}{4\sqrt{41}}=\frac{BD}{BC-BD}$ 05

$\frac{9}{4\sqrt{41}}=\frac{BD}{5-BD}$

$BD=\frac{45}{9+4\sqrt{41}}$ 05



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(c) $\cos^{-1}(1-x) = 2\sin^{-1}(x)$

$1 - x = \cos 2\sin^{-1}(x)$ (05)

$1 - x = 1 - 2\sin^2 \sin^{-1}(x)$

$1 - x = 1 - 2x^2$ (05)

$2x^2 - x = 0$

$x(2x-1) = 0$

$x = 0, x = 1/2$ (05)

15

අධ්‍යාපන, උසස් අධ්‍යාපන සහ වෘත්තීය අධ්‍යාපන අමාත්‍යාංශය

கல்வி, உயர் கல்வி மற்றும் தொழிற் கல்வி அமைச்சு

Ministry of Education, Higher Education and Vocational Education

Support Seminar – 2025

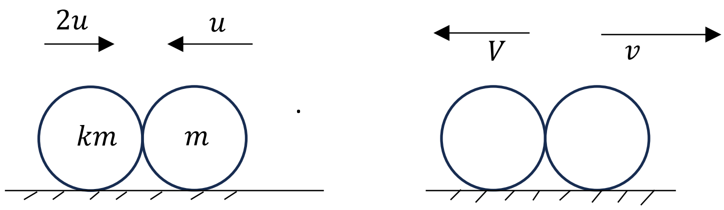
10 – Combined Mathematics

Paper II

Marking Scheme

Part A

1. Two particles A and B of masses km and m respectively are moving along a smooth horizontal table towards each other with speeds $2u$ and u respectively, collide directly. The coefficient of restitution between A and B is e. Just after the collision, if they move with equal speeds in the opposite directions, show that $e = \frac{2(2k-1)}{3(1-k)}$. **Hence** deduce that $\frac{1}{2} < k \leq \frac{5}{7}$



$Applying \quad I = \Delta mv \; for\ the\ system$

$$\rightarrow \quad 0 = mv - kmv - (2kmu - mu)$$ 5

$$(1-k)v = (2k-1)u$$

$$v = \left(\frac{2k-1}{1-k}\right)u$$

$Newton's\ Experiment\ law$

$$(v + v) = e(2u + u)$$ 5

$$3eu = \frac{2(2k-1)u}{1-k}$$

$$e = \frac{2(2k-1)}{3(1-k)}$$ 5

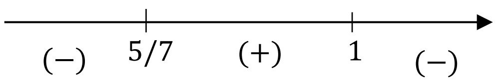
$$0 < e \leq 1$$

$$0 < \frac{2(2k-1)}{3(1-k)} \leq 1$$ 5

$$\frac{2(2k-1)}{3(1-k)} > 0 \Rightarrow \frac{1}{2} < k < 1$$

$$\frac{2(2k-1)}{3(1-k)} - 1 \leq 0$$

$$\frac{7k-5}{3(1-k)} \leq 0$$



$$\therefore \frac{1}{2} < k \leq \frac{5}{7}$$ 5

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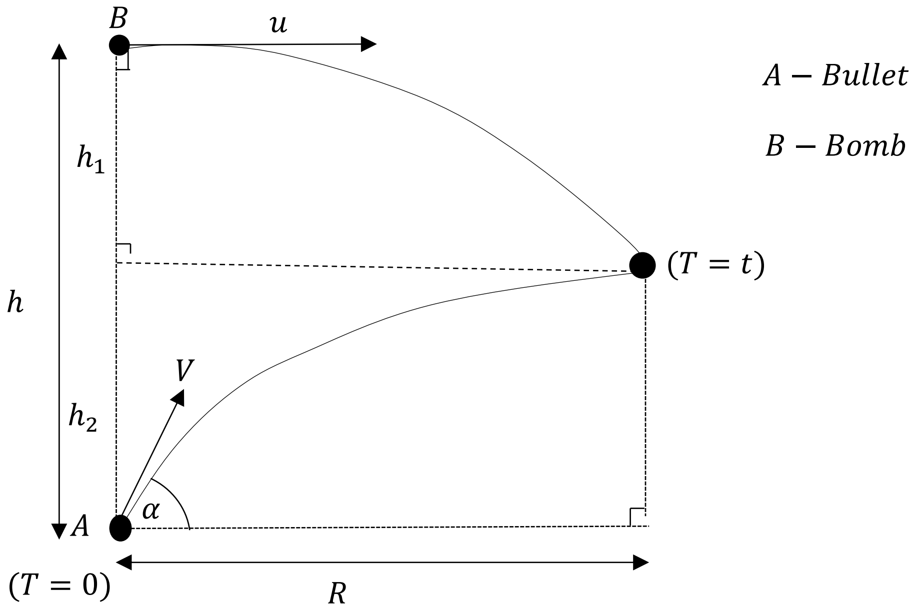
2. A bomb B is released from an aero-plane which is flying horizontally with the velocity u and at a constant height h. When it is released from rest and at the same time a bullet A is fired from a point which is vertically bellow the aero-plane. It is given that the bullet is fired at an angle θ to the horizontal to blast the bomb after time t. show that $t = \frac{h}{u}\ cot\theta$



$$S = ut + \frac{1}{2}at^2$$

$(B);\downarrow \quad h_1 = \frac{gt^2}{2} \longrightarrow (1)$ 5 $\qquad (A);\uparrow\ h_2 = V.\sin\alpha t - \frac{gt^2}{2} \longrightarrow (2)$ 5

$(B);\longrightarrow\ R = ut \longrightarrow (3) \qquad (A);\rightarrow\ R = V\cos\alpha.t \longrightarrow (4)$ 5

$(3),(4) \Rightarrow V = \frac{u}{\cos\alpha}$ 5

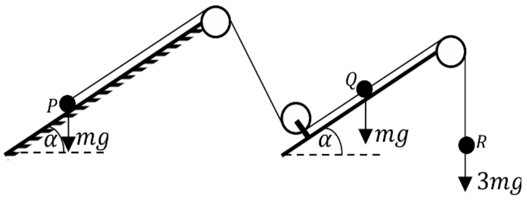
$$(1) + (2) \Rightarrow h_1 + h_2 = V\sin\alpha t$$

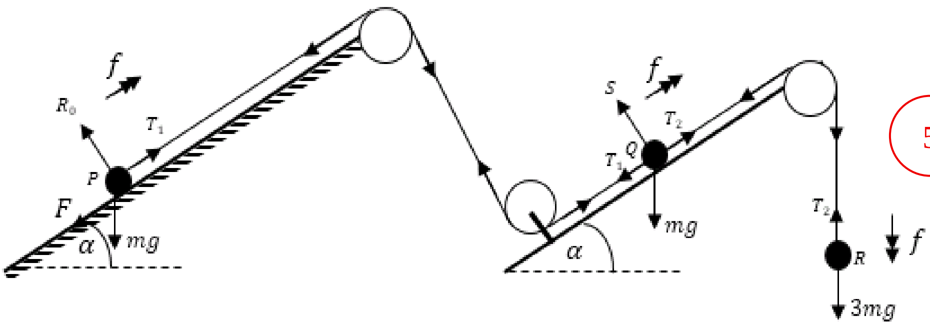
5

$$h = \sin\alpha.\frac{u}{\cos\alpha}.t \Longrightarrow \quad t = \frac{h}{u}.\cot\alpha$$

25

3. As shown in the figure, a particle P of mass m lies on a **rough** plane inclined at α to the horizontal. It is connected by a light inextensible string passing over two smooth pulleys to a particle Q of mass m on another **smooth** plane inclined at α to the horizontal. Another light inextensible string attached to Q, passes over a smooth pulley and is attached to a particle R of mass $3m$ hanging freely. The coefficient of friction between P and the plane is μ. The system is released from rest, and P moves up the plane. Write down the equations of motion sufficient to determine the tensions in the strings and the accelerations of the particles.





5 Forces & Accelerations

F = ma

For P $\quad T_1 - F - mg\,Sin\alpha = mf$ 5

↖ $\quad R_0 - mg\,Cos\alpha = 0$

$F = \mu R_0$ 5

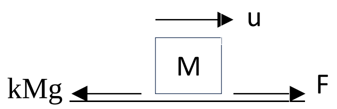
For Q ↗ $\quad T_2 - T_1 - mg\,Sin\alpha = mf$ 5

For R ↓ $\quad 3mg - T_2 = 3mf$ 5

25

4. A train mass M kg moves along a horizontal track at a uniform velocity. The rear compartment m kg (<M) of the train deatach and the engine driver is informed after it moves ℓ km after the detachment. At this moment the engine driver stops the engine. If the resistance to the motion of the train is uniform and proportional to the weight of the train then show that the distance between the rear compartment and the train is $\frac{M}{M-m}\ell$ when the both parts become at rest by considering the power exerted by the engine is constant. Assume that after the detachment of the compartment the engine continue the same tractive force until the driver disconnects the engine.

.

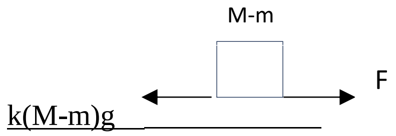


k is a constant

for train $\underline{F} = m\underline{a}$ $\longrightarrow$

$F - kMg = M.0$

$F = kMg$ (5)



For train (without last compartment)

$\longrightarrow$ $\underline{F} = m\underline{a}$

$F - k(M-m)g = (M-m)\,a$

$kMg - k(M-m)g = (M-m)\,a$

let acceleratem of the train , a

(5) $\frac{kmg}{M-m} = a$

For train

$\longrightarrow$ $v^2 = u^2 + 2as$

$w^2 = u^2 + 2\frac{kmg}{M-m}\ell$ (5)

for the train

$\longrightarrow$ $\underline{F} = m\underline{a}$

$-k(M-m)g = (M-m)\,a$

$-kg = a^2$

for train

$\longrightarrow$ $v^2 = u^2 + 2as$

$0 = u^2 + 2\frac{kmg}{M-m}\ell - 2kg\,S_1$

$S_1 = \frac{u^2}{2kg} + \frac{m}{M-m}\ell$ (5)

for the last compartment

$v^2 = u^2 + 2as$

$S_2 = \frac{u^2}{2kg}$ 5

Distance c between the train and last compartment = $S_1 - S_2 = \frac{m}{M-m}\ell$

25

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5. A particle P of mass m is attached to an end of a light inextensible string of length a and the other end of the string is tied to a fixed-point O. When the particle hangs freely a horizontal velocity of $\sqrt{ga}$ is given. Show that the tension in the string is $\frac{1}{2}mg$ when the particles comes to instantaneous rest.

$$(T.E)_B = (T.E)_A$$

$$-mga\cos\theta = \frac{1}{2}mga - mga$$ 15

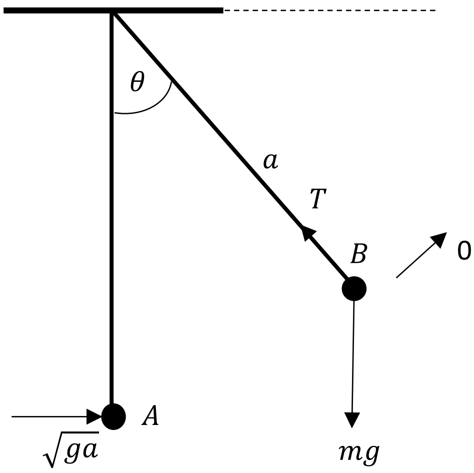
$$\cos\theta = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\nwarrow T - mg\cos\theta = 0$$ 10

$$T = mg \times \frac{1}{2}$$

$$= \frac{mg}{2}$$



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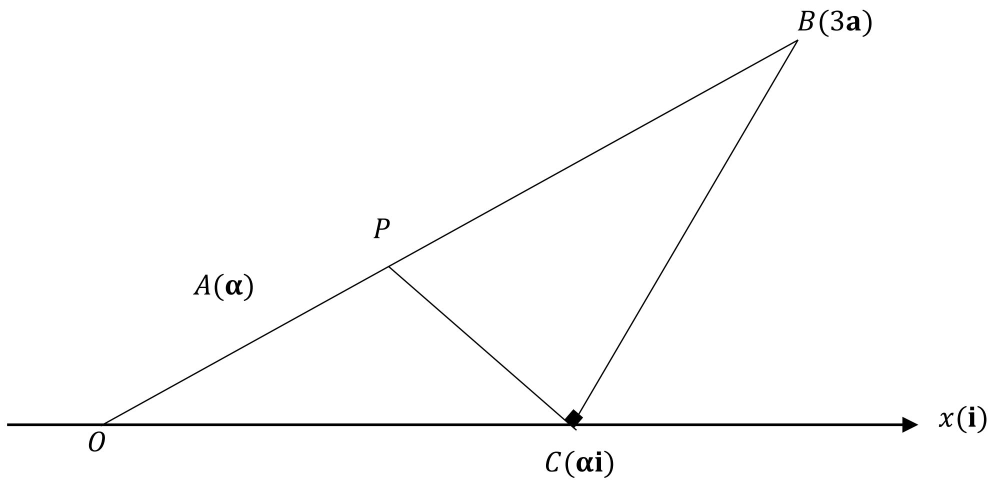
6. In usual notation, let the position vector of point A be $\boldsymbol{a} = 3\mathbf{i} + \mathbf{j}$. Point B has position vector $3\boldsymbol{a}$. O is the origin and P is the midpoint of OB. A point C lies on the x-axis such that $P\hat{C}B = \frac{\pi}{2}$. Show that $|\overrightarrow{OC}| = 6$ or $|\overrightarrow{OC}| = \frac{15}{2}$.



$$\overrightarrow{OP} = p = \frac{1}{2}\overrightarrow{OB} = \frac{3a}{2} = \frac{3}{2}(3\mathbf{i} + \mathbf{j})$$

Point C on the X − axis

$$\overrightarrow{OC} = \alpha\mathbf{i}$$ 5

$$\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \frac{3}{2}(3\mathbf{i} + \mathbf{j}) - \alpha\mathbf{i}$$

$$= \left(\frac{9}{2} - \alpha\right) i + \frac{3}{2} j$$

$$\overrightarrow{CB} = \mathbf{b} - \mathbf{c} = 3\mathbf{a} - \mathbf{c} = 3(3\mathbf{i} + \mathbf{j}) - \alpha\mathbf{j}$$

$$= (9 - \alpha)\mathbf{i} + 3\mathbf{j}$$ 5

$$\overrightarrow{CP} \perp \overrightarrow{CB} \Longrightarrow \overrightarrow{CP}.\overrightarrow{CB} = 0$$ 5

$$\left[\left(\frac{9}{2} - \alpha\right)\mathbf{i} + \frac{3}{2}\mathbf{j}\right].[(9 - \alpha)\mathbf{i} + 3\mathbf{j}] = 0$$

$$\left(\frac{9}{2} - \alpha\right)(9 - \alpha) + \frac{9}{2} = 0$$ 5

$$2\alpha^2 - 27\alpha + 90 = 0$$

$$(2\alpha - 15)(\alpha - 6) = 0$$

$$\therefore \alpha = \frac{15}{2} \; or \; \alpha = 6$$

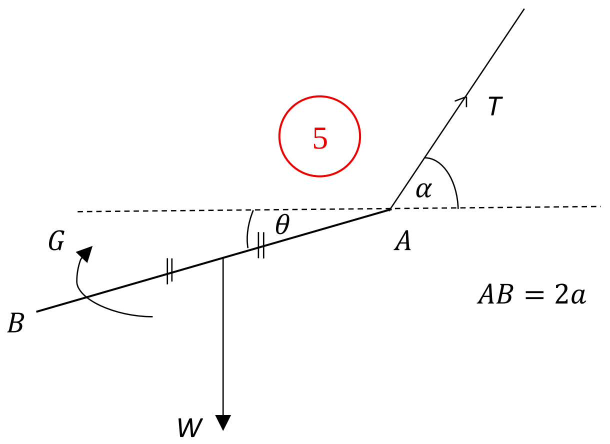
$$|\overrightarrow{OC}| = |\alpha\mathbf{i}| = \alpha$$ 5

$$\therefore |\overrightarrow{OC}| = 6$$

$$|\overrightarrow{OC}| = \frac{15}{2}$$

25

7. One end of a uniform rod of weight W and length $2a$ is attached to the end of a light string which hangs freely from a fixed point. A couple of moment G acts on the rod in a vertical plane containing the system. Find the tension in the string. Find the inclinations of the rod and the string when the system is in equilibrium. **Deduce** that $G \leq Wa$



$\rightarrow T\cos\alpha = 0$

$T \neq 0 \rightarrow \cos\alpha = 0 \Longrightarrow \quad \alpha = \frac{\pi}{2}$ (5)

$\uparrow T\sin\alpha - W = 0$

$T = W$ (5)

$A \circlearrowleft = 0$

$W.a\cos\theta - G = 0$

$\cos\theta = \frac{G}{Wa} \leq 1$ (5)

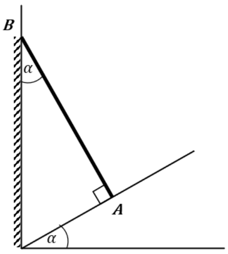
$G \leq Wa$

$\theta = \cos^{-1}\left(\frac{G}{Wa}\right)$ (5)

10 *marks should be given to the figure if string is drawn vertically*

25

8. A uniform rod AB of weight w and length $2a$ rests in equilibrium with A in contact with a rough plane inclined at α to the horizontal and B against a smooth vertical wall. The rod is perpendicular to the plane as shown in the diagram. If the coefficient of friction between the rod and rough plane is μ, then show that $\mu \geq \frac{\sin\alpha\cos\alpha}{2-sin^2\alpha}$.



For the equilibrium of the system

$B \circlearrowleft \quad Wa\sin\alpha - F.2a =$ (5)

$$F = \frac{W}{2}\sin\alpha$$

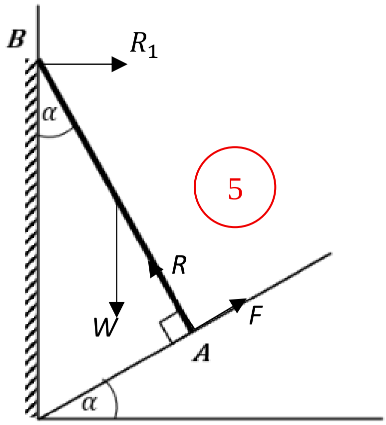
$\uparrow \quad R\cos\alpha + F\sin\alpha - W = 0$ (5)

$$R\cos\alpha = W - \frac{W}{2}\sin^2\alpha$$

$$R = \frac{W}{2}\left(\frac{2-\sin^2\alpha}{\cos\alpha}\right)$$ (5)

for equilibrium $\frac{F}{R} \leq \mu$ (5)

$$\frac{\sin\alpha\cos\alpha}{2-\sin^2\alpha} \leq \mu$$



(5)

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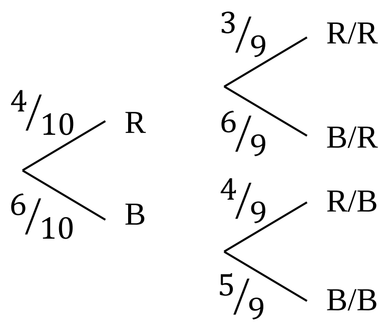
9. A bag contains 4 red balls and 6 blue balls which are equal in all respects, except for colour. Two balls are taken out at random without replacement.

(i) Find the probability that both balls are of the same colour.

(ii) Let Event A : "The first ball is red" and
Event B : "The second ball is red" .
Are the events A and B independent? Give reasons.

(i)



Probability that both balls are in same colour $= \frac{4}{10} \times \frac{3}{9} + \frac{6}{10} \times \frac{5}{9} = \frac{42}{90} = \frac{7}{15}$ (5)

(5)

Aliter Method

$P(R \cap R) + P(B \cap B) = P(R).\ P(R/R) + P(B).\ P(B/B)$

$= \frac{4}{10} \times \frac{3}{9} + \frac{6}{10} \times \frac{5}{9} = \frac{42}{90} = \frac{7}{15}$

(ii) $P(A) = \frac{4}{10} = \frac{2}{5}$

$P(B) = \frac{4}{10} . \frac{3}{9} + \frac{6}{10} . \frac{4}{9} = \frac{36}{90} = \frac{2}{5}$

$P(A \cap B) = \frac{4}{10} . \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$ (5)

$P(A)P(B) = \frac{2}{5} . \frac{2}{5} = \frac{4}{25}$ (5)

$P(A \cap B) \neq P(A)P(B)$. $\therefore$ A and B are not independent. (5)

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10. The marks a, b, c, and d obtained by four students are in ascending order. Where a, b, c, and d are positive integers. The mean of the marks is 37. The mean of the top three marks is 39. The range of marks is 15. Find the mean of the least three marks. If the mean deviation of the distribution is 5, then find b and c.

$a < b < c < d$

$\frac{a+b+c+d}{4} = 37 \Longrightarrow a + b + c + d = 148 \longrightarrow (1)$ (5)

$d - a = 15 \longrightarrow (2)$

$\frac{b+c+d}{3} = 39 \Rightarrow \quad b + c + d = 117 \longrightarrow (3)$ (5)

$(1) - (2)\ \ a = 148 - 117 = 31$

$(2) \Rightarrow \quad d = 15 + 31 = 46$

$\therefore \frac{a+b+c}{3} = \frac{148-46}{3} = 34$ (5)

$b + c = 148 - (46 + 31) = 71$

$\therefore c = 71 - b$

$mean\ deviation = 5 = |31 - 37| + |b - 37| + |71 - b - 37| + |46 - 37| = 5 \times 4 = 20$ (5)

$6 + 9 + |b - 37| + |b - 34| = 20$

$\therefore |b - 37| + |b - 34| = 5$

$Since\ b > 31, when\ b = 32, |32 - 37| + |32 - 34| \neq 5$

$Since\ b > 31, when\ b = 33, |33 - 37| + |33 - 34| = 5$

$b = 33$

$c = 38$ (5)

(25)

අධ්‍යාපන, උසස් අධ්‍යාපන සහ වෘත්තීය අධ්‍යාපන අමාත්‍යාංශය
கல்வி, உயர் கல்வி மற்றும் தொழிற் கல்வி அமைச்சு
Ministry of Education, Higher Education and Vocational Education

G.C.E.(A.L) Support Seminar 2025

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
Combined Mathematics II

10 S II

✱ Answer **five** questions only.

Part B

(In this question paper, g denotes the acceleration due to gravity.)

11. *(a)* A car P starts from rest at point O when time $t = 0$ s and moves with a constant acceleration f m s^{-2}. It accelerates uniformly until $t = T$ s and then continues with the constant speed it gained. Another car Q starts from rest at the same point O at $t = (T + 2)$ s and moves with the same constant acceleration f m s^{-2}. Q maintains this acceleration for $\frac{3T}{2}$ s and then continues with the constant speed it gained. Both cars move in the same direction on a straight horizontal road.

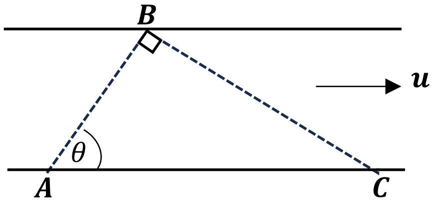
Sketch velocity-time graphs for the motions of P and Q in the same diagram. **Hence,**

(i) Find the distance between the cars P and Q in terms of f and T when their velocities are equal.

(ii) Show that Q **cannot** overtake P until Q gains its maximum speed.

(iii) Show that Q overtakes P at $t = \frac{17T+24}{4}$ s.

(b) A river with two straight parallel banks flows with the constant velocity u. The points A, B and C are on the river banks such that $A\hat{B}C = \frac{\pi}{2}$.



It is also given that $AC = a$ and $B\hat{A}C = \theta$; $\left(0 < \theta < \frac{\pi}{2}\right)$

(see the adjoining figure)

A swimmer can swim with a constant speed $2u$ in still water, starts at A and swims along a straight line to reach B. Then he swims along a straight line to reach C. Assume that no time taken for turning.

By constructing the velocity triangle for the motion from A to B, or **otherwise**,

show that the magnitude of the velocity of swimmer along AB is $(\sqrt{3 + cos^2\theta} + cos\theta)u$.

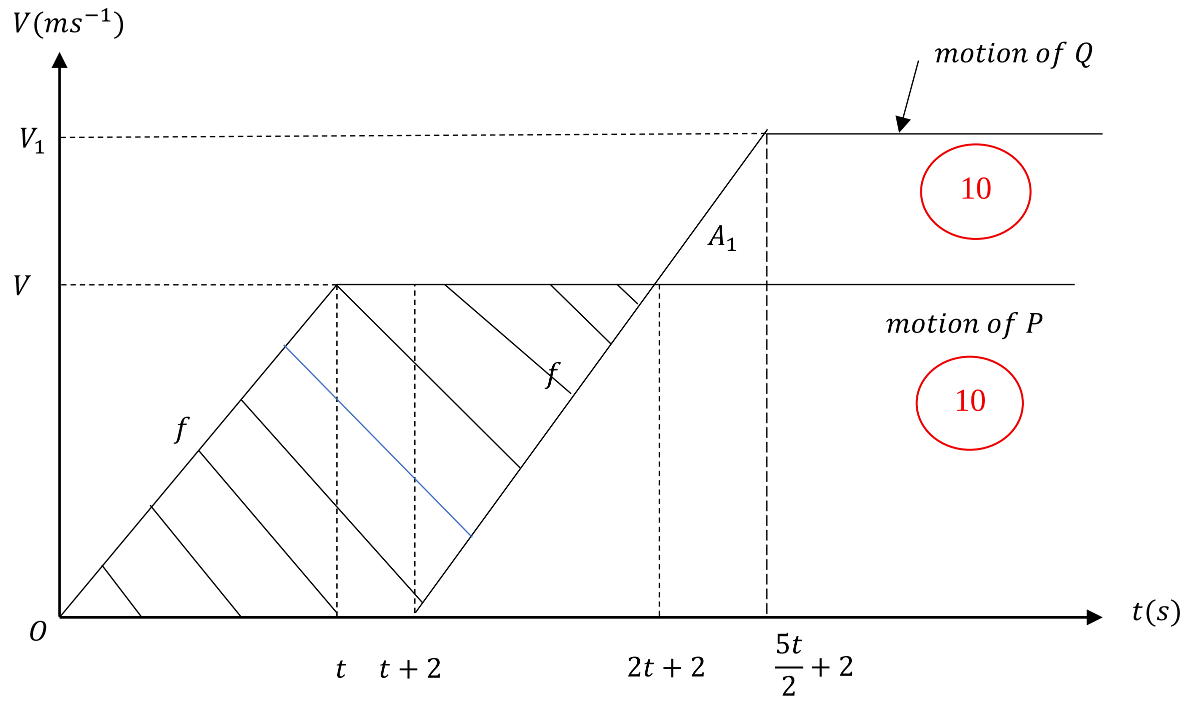
Deduce the magnitude of velocity along BC.

Also show that the total time taken by the swimmer to reach C from A is

$\frac{a}{3u}(cos\theta\sqrt{3 + cos^2\theta} + \ sin\theta \ \sqrt{3 + sin^2\theta} \ - 1)$

Further, **deduce** the value of θ such that the two time periods along AB and BC are same.

11(a)



(i) *when the velocities are equal*

$$T = 2t + 2$$

$\therefore$ *Distance travelld by* $(P - Q)$

$$= \frac{1}{2} \times \text{shaded area}$$

$$= \frac{1}{2} \times (2ft + 2) \times V \quad (5)$$

$$\text{but } f = \frac{V}{t} \Longrightarrow V = ft \quad (5)$$

$$\therefore \text{Distance} = \frac{2}{2}(t+2)ft = (t+2)ft \quad (5)$$

35

(ii). *When Q reaches its maximum Distance between P and Q*

$$= \text{Previous area} - A_1 \quad (5)$$

$$= (t+2)ft - \frac{1}{2} \times \frac{t}{2}\left(\frac{3ft}{2} - ft\right) \quad (5)$$

$$= (t+2)ft - \frac{ft^2}{8} \quad (5)$$

$$= \frac{ft}{8}(16 + 7t) \quad (5)$$

this is always > 0 (5)

$\therefore$ *Q cannot Overtakes P until it reaches its maximum* speed. (5)

30

(iii). *Let the time taken to Q to Over take P after* $(2t+2)s$ *is* t_0 $\therefore$ *when Overtake P,* 5

5

$$ft(t+2) = \frac{1}{2} \times \left(t_0 + t_0 - \frac{t}{2}\right) \times \frac{ft}{2}$$ 5

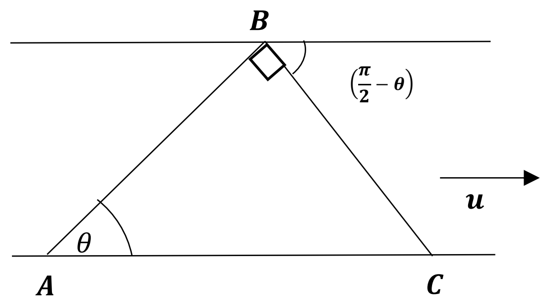
$$8(t+2) = (4t_0 - t)$$ 5

$$4t_0 = 9t + 16$$

$$t_0 = \left(\frac{9t+16}{4}\right)s$$ 5

25

(b)

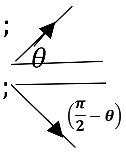


$W - Water$

$E - Earth$

$S - swimmer$

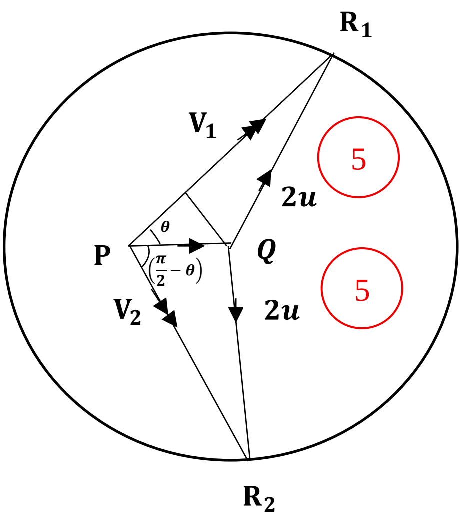
$$V_{(S.E)} = V_{(S.W)} + V_{(W.E)}$$

$AB;$ 

$= 2u + \vec{u}$ 5

$BC;$ $= 2u + \vec{u}$ 5

10



5

5

$PQR_1\Delta;\ V_{AB} = V_1$

$$= u\cos\theta + \sqrt{(2u)^2 - (u\sin\theta)^2}$$ 5

$$= u\left(\cos\theta + \sqrt{3+\cos^2\theta}\right)$$

$$V_{BC} = (V_{AB})_{\theta \to \left(\frac{\pi}{2}-\theta\right)}$$ 5

$$= u\left(\cos\left(\frac{\pi}{2} - \theta\right) + \sqrt{3 + \cos^2\left(\frac{\pi}{2} - \theta\right)}\right) = u\left(\sin\theta + \sqrt{3 + \sin^2\theta}\right)$$ 5

25

$$t_{AC} = t_{AB} + t_{BC}$$

$$= \frac{a\cos\theta}{V_1} + \frac{a\sin\theta}{V_2}$$ 5

$$= \frac{a\cos\theta}{u\left(\cos\theta + \sqrt{3+\cos^2\theta}\right)} + \frac{a\sin\theta}{u\left(\sin\theta + \sqrt{3+\sin^2\theta}\right)}$$ 5

$$= \frac{a}{u}\left[\frac{\cos\left(\sqrt{3+\cos^2\theta}\right) - \cos\theta}{3} + \frac{\sin\sqrt{3+\sin^2\theta} - \sin\theta}{3}\right]$$ 5

$$= \frac{a}{3u}\left(\cos\theta.\sqrt{3+\cos^2\theta} + \sin\left(\sqrt{3+\sin^2\theta} - 1\right)\right)$$

$$t_{AB} = t_{BC}$$

$$\frac{a\cos\theta}{u\left(\cos\theta + \sqrt{3+\cos^2\theta}\right)} = \frac{a\sin\theta}{u\left(\sin\theta + \sqrt{3+\sin^2\theta}\right)}$$ 5

$when\ \theta = \frac{\pi}{4}\ ; L.H.S = R.H.S$

$\therefore\ \theta = \frac{\pi}{4}$ 5

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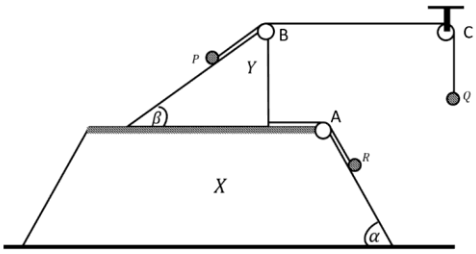
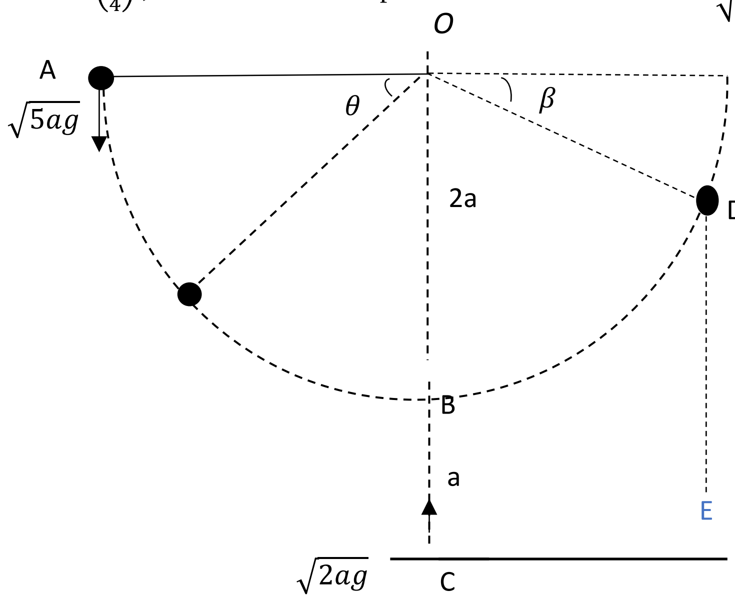
12. (*a*) The figure shows a vertical cross section passing through the centers of mass of two uniform wedges X and Y and three particles P, Q and R. A wedge Y of mass M is placed on the **rough** horizontal face of the fixed wedge X. α and β are the inclinations of the smooth inclined planes of X and Y respectively. A and B are two small smooth pulleys fixed to the wedges X and Y respectively. C is a fixed pulley in the same vertical plane such that BC is horizontal. Two particles P and Q of masses m_1 and m_2 are attached to the ends of a light inextensible string passing over the pulleys B and C. Another light inextensible string passing over the pulley A connected to Y to the particle R of mass m_3 which is on the inclined face of X. The coefficient of friction between X and Y is μ. The system is released from rest with the strings taut. Assuming that during the entire motion of the particle P, is in contact with the inclined plane of Y, write equations sufficient to determine the accelerations of the particles and the wedges.



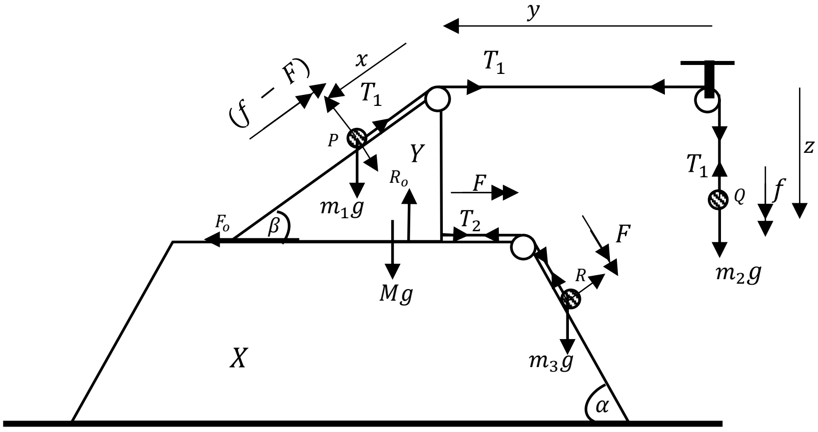
(*b*) One end of a light inextensible string of length $2a$ is attached to a fixed-point O, and the other end to a Particle P_1 of mass m. The particle is held at A which is same level of O, such that $OP_1 = 2a$ and projected vertically downwards with speed $\sqrt{5ag}$. Show that the speed V of the Particle P_1 when OP_1 makes an angle θ $\left(0 < \theta < \frac{\pi}{2}\right)$ with horizontal is given by $V^2 = ag(5 + 4\ \sin\theta\ )$ and find the tension in the string. When P_1 reaches its lowest point B, it collides directly in the horizontal direction with a particle P_2 of mass m which has been projected vertically upwards with a speed $\sqrt{2ag}$ from a point at a depth of $3a$ below O. In the motions of P_1, P_2 after the collision, P_1 continues circular motion and comes to a momentary rest at point D, which is the angle OD is β with horizontal. The particle P_2 moves under gravity and reaches E which is the point vertically below D.

(i) If the coefficient of restitution between P_1 and P_2 is $e = \frac{1}{3}$, show that the velocity of P_1after collide is $\sqrt{ag}$.

(ii) Show that $\beta = \sin^{-1}\left(\frac{3}{4}\right)$, Also show that the particle reaches E when $t = \sqrt{\frac{7a}{16g}}$ sec, after impact.



(a)



$$x + y + z = l$$
$$\ddot{x} + \ddot{y} + \ddot{z} = 0$$
$$\ddot{x} = -(\ddot{y} + \ddot{z})$$ [5]

$a_{(Y,E)} = F \rightarrow$

$a_{(P,Y)} = (f - F)$ ↗ (angle β)

$a_{(Q,E)} = f \downarrow$

$a_{(R,E)} = F$ ↘ (angle α) [5]

$a_{(P,E)} =$ $(f - F)$ ↗ (angle β) + $F \rightarrow$ [5]

$\underline{F = ma}$

For Q, ↓ $m_2 g - T_1 = m_2 f$ [10]

For P, ↗ (angle β) $T_1 - m_1 g \sin\beta = m_1(f - F + F\cos\beta)$ [10]

For R, ↘ (angle α) $m_3 g \sin\alpha - T_2 = m_3(F)$ [10]

For the system (Y and P),

→ $T_1 + T_2 - F_0 = MF + m_1((f - F)\cos\beta + F)$ [15]

↑ $R_o - Mg - m_1 g = m_1(f - F)\sin\beta$ [10]

For Y, $F_0 = \mu R_o$ [5]

[75]

(b). By the conservation of energy, we have

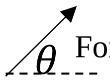
$$(T.E)_A = (T.E)_B$$

$$\frac{1}{2}m.5ag + 0 = \frac{1}{2}mV^2 - mg.2a\sin\theta \quad (10)$$

$$5ag = V^2 - 4ag\sin\theta$$

$$V^2 = ag(5 + 4\sin\theta) \quad (5)$$

15



For particle P_1: apply $F = Ma$

$$T - mg\sin\theta = m\frac{V^2}{2a}$$

$$T = mg\sin\theta + m\frac{ag(5 + 4\sin\theta)}{2a} \quad (10)$$

$$T = \frac{mg}{2}(6\sin\theta + 5) \quad (5)$$

When $\theta = \frac{\pi}{2}$ velocity of P_1

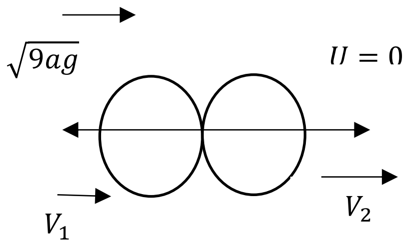
$$V^2 = ag\left(5 + 4\sin\frac{\pi}{2}\right)$$

$$= 9ag$$

$$V = 3\sqrt{ag} \quad (5)$$

Velocity of P_2 $\quad V^2 = U^2 + 2ga$

$$V = 0$$



$$V_2 - V_1 = \frac{1}{3}.3\sqrt{ag} \quad (5)$$

$$V_1 + V_2 = 3\sqrt{ag} \quad (5)$$

$$V_1 = \sqrt{ag} \quad (5)$$

35

After impact By the conservation of energy, we have

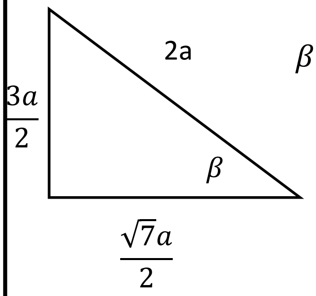
$$(T.E)_B = (T.E)_D$$

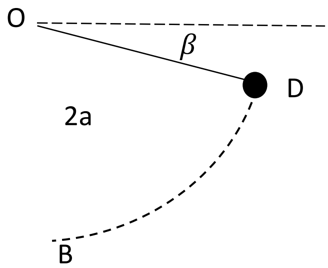
$$\frac{1}{2}m.ag + mg(-2a) = 0 - mg.h$$ 10

$$h = \frac{3a}{2}$$ 5

$$sin\, sin\, \beta\ = \frac{3}{4}$$

$$\beta = \left(\frac{3}{4}\right)$$





For particle P_1 from $\overrightarrow{BE}$

$$s = ut + \frac{1}{2}gt^2$$

$$2a\, cos\, cos\, \beta\ = V_2 t$$

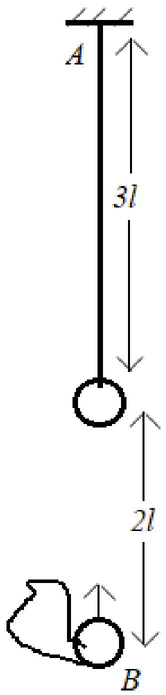
$$t = \frac{2a\, cos\, cos\, \beta}{V_2}$$ 10

$$t = \frac{2.a.\frac{\sqrt{7}a}{2}}{2\sqrt{ag}}$$

$$t = \sqrt{\frac{7a}{16g}}$$

25

13. A particle Q of mass m is suspended by a light elastic string of natural length $2l$ and modulus of elasticity λ from a fixed point A, which is at height $5l$ vertically above another fixed point B. When Q is in equilibrium, the length of the string is $3l$. Show that $\lambda = 2mg$. A particle P of mass m is attached to the end of another light elastic string of natural length l_0 and modulus of elasticity $2mg$, the other end is attached to B. P is projected vertically upwards with a speed $\sqrt{6gl}$. If P just reaches Q, by applying principle of conservation of energy, show that $l_0 = l$. Find the velocity of P when $BP = l$.



When the length of the string BP is $(l + x)$, show that the equation of motion of P is given by $\ddot{x} + \frac{2g}{l}\left(x + \frac{l}{2}\right) = 0$. Let $x + \frac{l}{2} = X$, show that $\ddot{X} + \omega^2 X = 0$ where ω is a real constant to be determined. Find the centre and amplitude of the simple harmonic motion of P. When the particle P reaches Q, it sticks with Q. When the composite particle S is at a height $(l + y)$ above B, show that the equation of motion of S is $\ddot{y} + \left(\frac{3g}{2l}\right) y = 0$.

Find the centre and amplitude of the simple harmonic motion of S. Also Show that the total time taken for the lower string to become slack for the first time from the time that P was projected is $\sqrt{\frac{l}{6g}}\left(\pi + 6 - 2\sqrt{6} + \sqrt{3}\cos^{-1}\left(\frac{1}{3}\right)\right)$

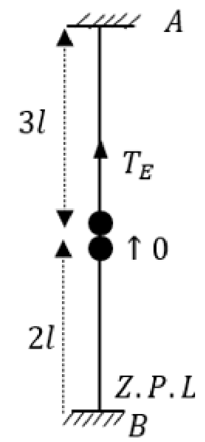
For Q

$\uparrow \quad T_E - mg = 0$ (5)

$\frac{\lambda l}{2l} = mg$

$\lambda = 2mg$ (5)

[10]



For (P), form B to reach (Q)

P.C.E

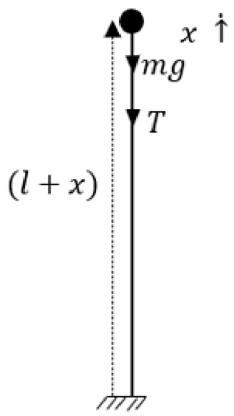
$$0 + \frac{2mg}{2l_0}(2l - l_0)^2 + mg.2l = \frac{1}{2}m.6gl$$ (20)

$$(2l - l_0)^2 = ll_0$$

$$l_0^2 - 5ll_0 + 4l^2 = 0$$

$$(l_0 - 4l)(l_0 - l) = 0$$ (5)

$$0 < l_0 < 2l \Rightarrow l_0 = l$$ (5)



For (Q)

$\uparrow \quad v^2 = u^2 + 2as$

$$v_0^2 = 6gl - 2gl$$

$$v_0 = 2\sqrt{gl}$$ (5)

[35]

Applyin **F = ma** $for\ (P) \uparrow$

$$-T - mg = m\ddot{x}$$ 5

5 $$-\frac{2mgx}{l} - mg = m\ddot{x}$$

$$\ddot{x} + \frac{2g}{l}\left(x + \frac{l}{2}\right) = 0$$

$$x + \frac{l}{2} = X$$

$Differentiate\ w.r.t.\ \boldsymbol{t}$

$$\dot{x} = \dot{X}$$ 5

$$\ddot{x} = \ddot{X}$$

$$\ddot{X} + \omega^2 X = 0 \Longrightarrow \omega = \sqrt{\frac{2g}{l}}$$ 5

$$\therefore SHM$$

$$Centre;\ \ddot{x} = 0$$

$$x = 0$$

$$x = -\frac{l}{2}$$

centre is at a height $\frac{l}{2}$ **above B** 5

$\dot{\mathbf{x}} = \mathbf{0}$ **at a height 2l above B**

$\therefore$ **amplitude** $= 2l - \frac{l}{2}$

$$= \frac{3l}{2}$$ 5

30

Applying F = ma For (S) $\uparrow$

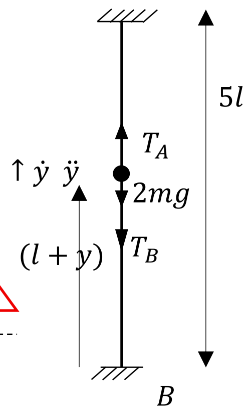
$$T_A - T_B - 2mg = 2m\ddot{y}$$ [10]

[10] $$\frac{2mg}{2l}(4l - y - 2l) - \frac{2mgy}{l} - 2mg = 2m\ddot{y}$$

$$g - \frac{gy}{2l} - \frac{gy}{l} - g = \ddot{y}$$

$$\ddot{y} + \frac{3g}{2l}y = 0$$ [5]

[25]



$$\therefore SHM \Longrightarrow \omega_1 = \sqrt{\frac{3g}{2l}}$$ [5]

centre ; $\ddot{y} = 0 \Longrightarrow y = 0$

[5] **centre is at a height** l **above B**

amplitude $= 2l - l = l$ [5]

[15]

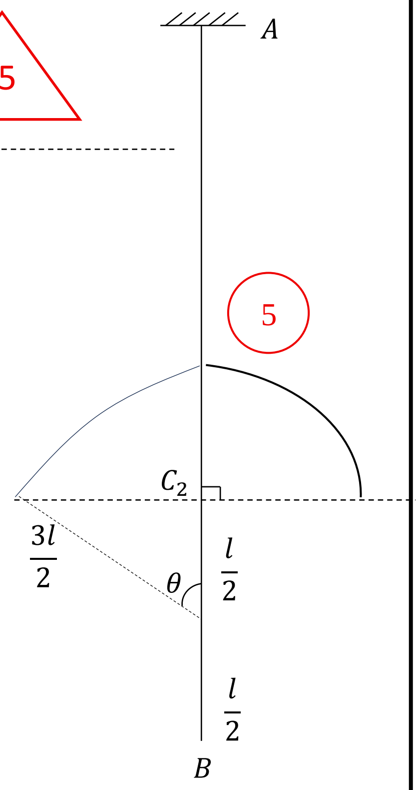
$$\cos\theta = \frac{l/2}{3l/2} = \frac{1}{3}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$ [5]

For $(P) \uparrow v = u + at$

$$2\sqrt{gl} = \sqrt{6gl} - gt_0$$ [5]

$$t_0 = (\sqrt{6} - 2)\sqrt{\frac{l}{g}}$$



[5] [5] [5]

$$t = \frac{\theta}{\omega} + \frac{\pi}{2\omega_1} + (\sqrt{6} - 2)\sqrt{\frac{l}{g}}$$

$$= \sqrt{\frac{l}{6g}}\cos^{-1}\left(\frac{1}{3}\right) + \frac{\pi}{2}\sqrt{\frac{2l}{3g}} + (\sqrt{6} - 2)\sqrt{\frac{l}{g}}$$ [5]

$$= \sqrt{\frac{l}{6g}}\left(\sqrt{3}\cos^{-1}\left(\frac{1}{3}\right) + \pi + 6 - 2\sqrt{6}\right)$$

[35]

---

14. (*a*) A, B, C and D are four distinct points on the oxy-plane. Position vectors of A and B are $\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} + 9\mathbf{j}$ respectively, in usual notation. Also $\overrightarrow{BC} = 2\mathbf{i} - 3\mathbf{j}$ and $\overrightarrow{BD} = 6\mathbf{i} - 9\mathbf{j}$. Show that the points O, A and B are colinear. The lines OC and AD meet at E. For a positive constant α, $OE : OC = \alpha : 1$. Show that $\boldsymbol{e} = 5\alpha\mathbf{i} + 6\alpha\mathbf{j}$ where $\boldsymbol{e}$ is the position vector of the point E with respect to the vector origin. For another positive constant β, $AE : AD = \beta : 1$. Obtain another expression for $\boldsymbol{e}$ in terms of β.

Hence, show that $\alpha = \frac{3}{7}$ and $\beta = \frac{1}{7}$. Further, show that $O\hat{B}D = Cos^{-1}\left(\frac{7}{\sqrt{130}}\right)$.

(*b*) $ABCDEG$ is a regular hexagon of side a and O is the centre. Forces of magnitudes $2, 3, 4, 5$ units act along AB, BC, CD, DE respectively and a variable force $\boldsymbol{F}$ acts at G in the plane of the hexagon. Cartesian axes OX, OY are chosen parallel to AB, BD respectively. The resultant of the system of forces will be denoted by $\boldsymbol{R}$.

(i) If $F = 0$ show that $|R| = 7$ and find the Cartesian equation of the line of action of R.

(ii) Find the magnitude and direction of F, indicating the direction clearly in a figure. If the line of action of R is to coincide with OD, find $|R|$.

(iii) If F is chosen to make the system of forces reduce to a couple, show that the moment of the couple is of magnitude $\frac{21\sqrt{3}\,a}{2}$. Now, if the force acting on AB is removed, deduce the resultant of the new system of forces.

(*a*) $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j}, \overrightarrow{OB} = 3\mathbf{i} + 9\mathbf{j}, \overrightarrow{BC} = 2\mathbf{i} - 3\mathbf{j}, \overrightarrow{BD} = 6\mathbf{i} - 9\mathbf{j}$

$\overrightarrow{OB} = 3(\mathbf{i} + 3\mathbf{j}) = 3\overrightarrow{OA} \Longrightarrow O, A, B$ are collinear. (5) (5)

$OE : OC = \alpha : 1$

$\overrightarrow{OE} = \alpha\overrightarrow{OC}$ (5)

$= \alpha(OB + BC)$

$= \alpha(3\mathbf{i} + 9\mathbf{j} + 2\mathbf{i} - 3\mathbf{j})$

$= 5\alpha\mathbf{i} + 6\alpha\mathbf{j} \quad \longrightarrow (1)$ (5)

$AD = \beta : 1$

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$ (5)

$= \overrightarrow{OA} + \beta\overrightarrow{AD}$ (5)

$= \mathbf{i} + 3\mathbf{j} + \beta(\overrightarrow{AB} + \overrightarrow{BD})$

$= \mathbf{i} + 3\mathbf{j} + \beta(2\mathbf{i} + 6\mathbf{j} + 6\mathbf{i} - 9\mathbf{j})$ (5)

$= \mathbf{i} + 3\mathbf{j} + \beta(8\mathbf{i} - 3\mathbf{j})$

$= (8\beta + 1)\mathbf{i} + (3 - 3\beta)\mathbf{j} \quad \longrightarrow (2)$ (5)

$(1) = (2)$

$8\beta + 1 = 5\alpha$

$3 - 3\beta = 6\alpha$ (5)

$\frac{8\beta + 1}{3 - 3\beta} = \frac{5}{6}$

$48\beta + 6 = 15 - 15\beta$

$\beta = \frac{9}{63} = \frac{1}{7}$ (5)

$\therefore 3 - \frac{3}{7} = 6\alpha = \frac{18}{7}$

$\therefore \alpha = \frac{3}{7}$ (5)

55

Let $O\hat{B}D = \theta$

$$\overrightarrow{BO} = -3\mathbf{i} - 9\mathbf{j}$$

$$\overrightarrow{BD} = 6\mathbf{i} - 9\mathbf{j}$$

$$\overrightarrow{BO}.\overrightarrow{BD} = |\overrightarrow{BO}||\overrightarrow{BD}| \cos\theta$$ 5

$$(-3\mathbf{i} - 9\mathbf{j}).(6\mathbf{i} - 9\mathbf{j}) = \sqrt{3^2 + 9^2}\sqrt{6^2 + 9^2} \cos\theta$$ 5

$$-18 + 81 = \sqrt{90}\sqrt{117} \cos\theta$$

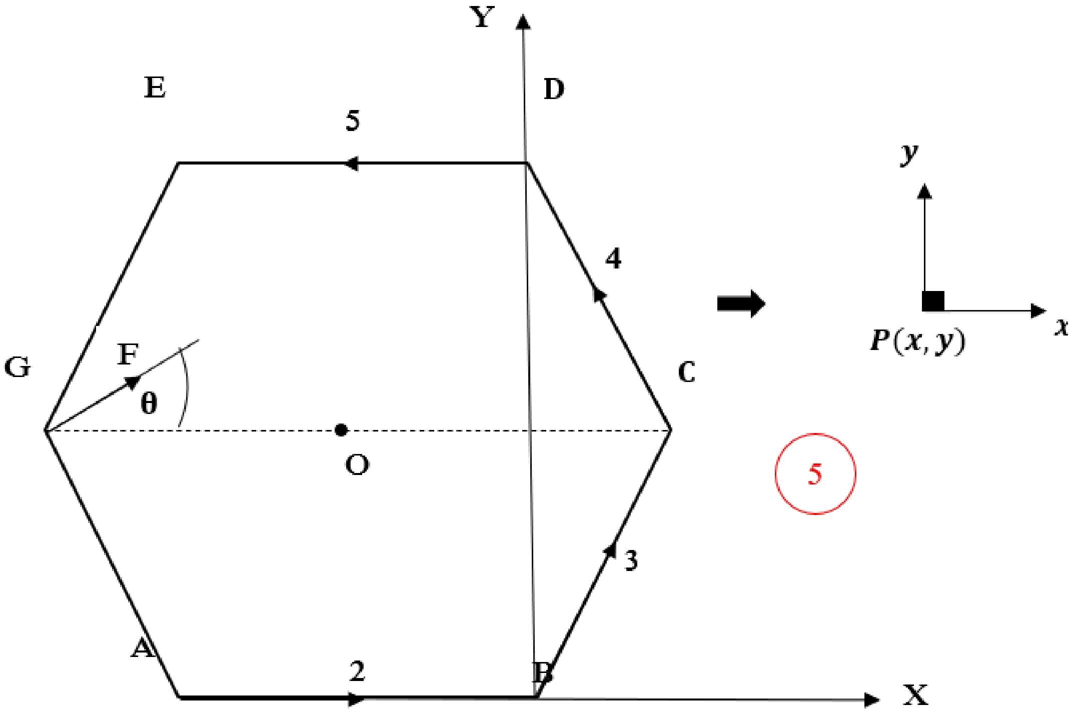
5

$$\frac{63}{\sqrt{3.30}\sqrt{3.39}} = \cos\theta = \frac{63}{3\sqrt{3.10}\sqrt{3.13}} = \frac{7}{\sqrt{130}}$$ 5

$$\theta = \cos^{-1}\left(\frac{7}{\sqrt{130}}\right)$$

20

(b)



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i) $If \quad \underline{F} = \underline{O}$

$\rightarrow X = 2 + 3\cos 60° - 4\cos 60° - 5$

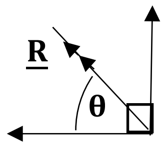
$= -3 - \cos 60°$

$= -\frac{7}{2}$ 5

$\uparrow Y = 3\cos 30° + 4\cos 30°$

$= 7\cos 30°$

$= \frac{7\sqrt{3}}{2}$ 5



$$R^2 = \left({}^7/_2\right)^2 + \left({}^{7\sqrt{3}}/_2\right)^2 = 7^2$$

$R = 7 \implies |R| = 7$ 5

$\overset{\curvearrowleft}{B}\ Res = \overset{\curvearrowleft}{B}\ Sys$

$Y.x - X.y = 5.2a\cos 30° + 4.a\cos 30°$

$\frac{7\sqrt{3}}{2}x + \frac{7}{2}y = 14a\frac{\sqrt{3}}{2}$

$y + \sqrt{3}x = 2\sqrt{3}a$ 5

25

$(ii) \rightarrow X = F\cos\theta - {}^7/_2 > 0$

$\uparrow Y = F\sin\theta + {}^{7\sqrt{3}}/_2 > 0$

$\frac{Y}{X} = Tan\,60° \implies Y = \sqrt{3}X$ 5

$F\sin\theta + {}^{7\sqrt{3}}/_2 = \sqrt{3}\left(F\cos\theta - {}^7/_2\right)$ (1)

since **R goesthrough *O*,**

$\overset{\curvearrowleft}{O}\ Sys = 0$

$(2+3+4+5)a\cos 30° - F\sin\theta.a = 0$

$F\sin\theta = {}^{14\sqrt{3}}/_2$

$F\sin\theta = 7\sqrt{3}$ 5

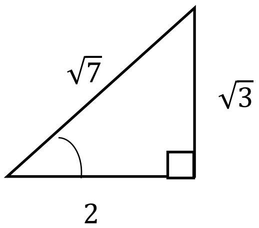
$① \Rightarrow 7\sqrt{3} + {}^{7\sqrt{3}}/_2 = \sqrt{3}\left(F\cos\theta - {}^7/_2\right)$

$2\,{}^1/_2 = F\cos\theta - {}^7/_2$

$F\cos\theta = 14$ 5

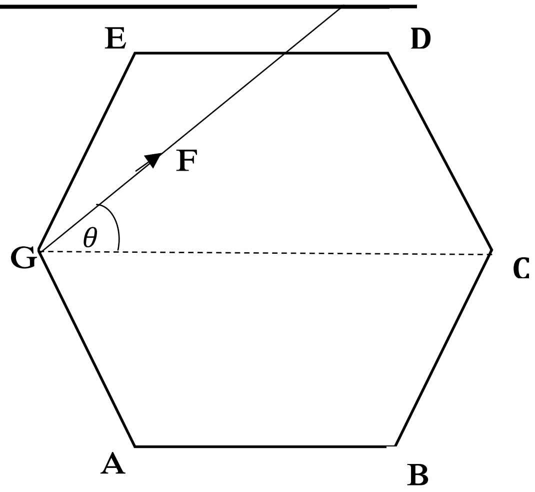
$\frac{F\sin\theta}{F\cos\theta} = \frac{7\sqrt{3}}{14} \implies Tan\,\theta = {}^{\sqrt{3}}/_2$

$\theta = \mathrm{Tan}^{-1}\,{}^{\sqrt{3}}/_2$ 5



$\mathrm{F}\sin\theta = 7\sqrt{3}$

$F.\frac{\sqrt{3}}{\sqrt{7}} = 7\sqrt{7}\,units.$



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(iii) $\rightarrow F\cos\theta - 7/2 = 0 \Rightarrow F\cos\theta = 7/2$ —— ② 5

$\uparrow F\sin\theta + 7\sqrt{3}/2 = 0 \Rightarrow F\sin\theta = -7\sqrt{3}/2$ —— ③ 5

②, ③ $\Rightarrow Tan\,\theta = -\sqrt{3} \Rightarrow \theta = -60°$

② $\Rightarrow F\cos(-60°) = 7/2$

$F = 7$ 5

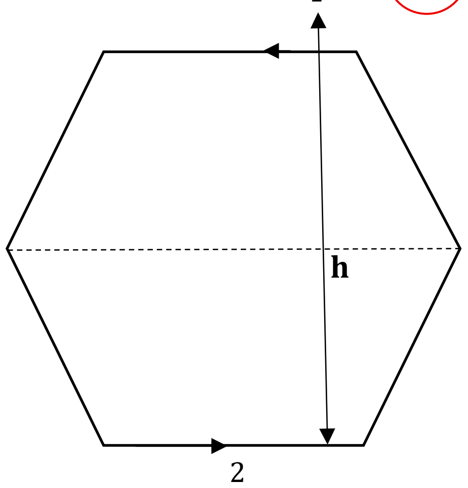
F is 7 units along GA

Moment of the Couple

$= 0 \quad Sys$

$= (2+3+4+5+7)a\cos 30°$

$= \frac{21\sqrt{3}a}{2}$ 5



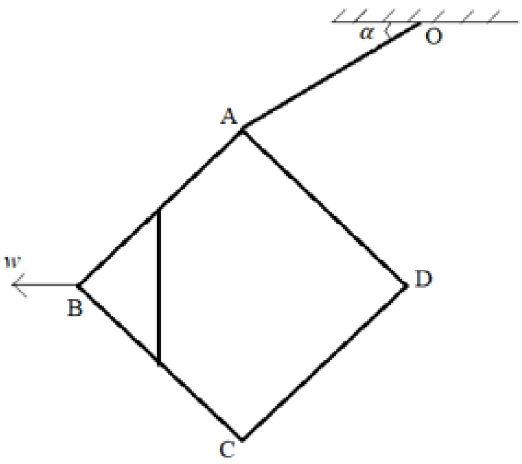
$2h = 21\frac{\sqrt{3}}{2}a$

$h = 21\frac{\sqrt{3}}{4}a$

∴ If the fore acting along AB is removed resultant of the new system is of magnitude 2 Units to the direction of BA at a height of $\frac{21\sqrt{3}a}{4}$ form AB (x-axis). 5

25

15. (a) Three uniform rods AB, CD and AD, each of equal length $2a$ and equal weight w, and another uniform rod BC of weight λw and of length $2a$, are jointed smoothly at their ends. The framework is suspended from a fixed point by a light inextensible string OA. A light rod jointed to the midpoints of AB and BC to maintain the framework in the shape of a square. It is kept in equilibrium with AC vertical, as shown in the adjoining figure, by applying a horizontal force of magnitude w at B.



(i) Show that the horizontal and vertical components of the reaction at C on the rod BC are $\frac{w}{2}$ and w respectively.

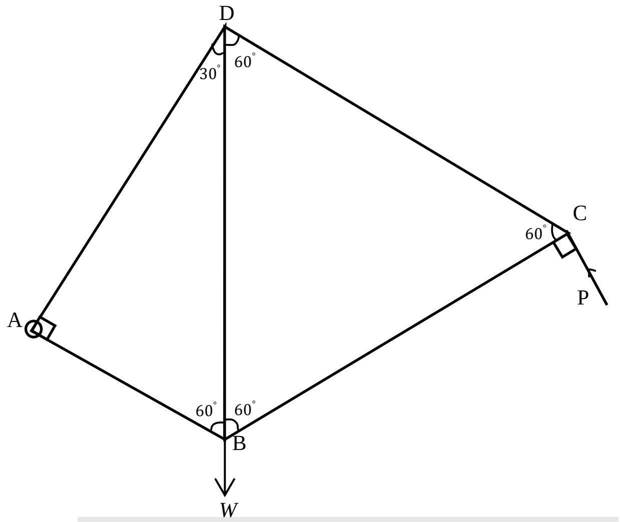
(ii) Show that $\lambda = 3$

(iii) Calculate the stress in the light rod.

(b) Five light rods freely jointed to form a framework in which $2AB = BC = BD = DC$, and $B\hat{A}D = 90°$. The framework is smoothly hinged to a fixed point at A and kept in equilibrium in a vertical plane. A weight w is suspended from B and a force P is applied at C, perpendicular to BC in the same plane of the frame as shown in the figure.

Show that $P = \frac{\sqrt{3}}{5}w$.

Draw the stress diagram using Bow's notation and find the stresses in each rods, stating whether they are tensions or thrusts.



($\mathbf{a}$) (i) $for\ AD\ and\ CD$

$A \circlearrowleft \quad X_1 4a\cos 45^0 - 2Wa\cos 45^0 = 0$ 10

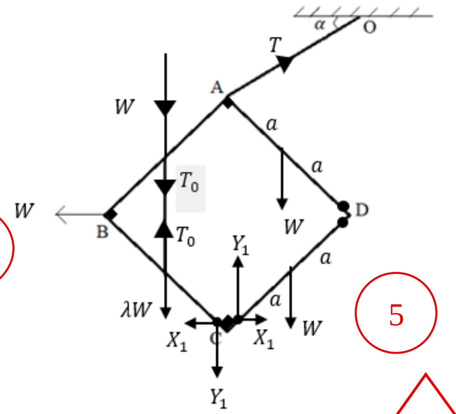
$X_1 = \frac{W}{2}$ 5

$For\ CD$

$D \circlearrowright Y_1 . 2a\cos 45^0 - X_1 2a\sin 45^0 - Wa\cos 45^0 = 0$ 10

$2Y_1 = W + W$

$Y_1 = W$ 5



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(ii) *For AB and BC*

$A \circlearrowleft \; W(1+\lambda)a\cos 45^0 - X_1 4a\sin 45^0 - W2a\cos 45^0 = 0$ (10)

$W(1+\lambda) = 2W + 2W$ (5)

$1 + \lambda = 4 \Longrightarrow \lambda = 3$

15

(iii) For BC, $B \circlearrowleft \quad T_0 a\cos 45^0 - 3W\,a\cos 45^0 - \frac{W}{2}2a\sin 45^0 - W2a\sin 45^0 = 0$ (10)

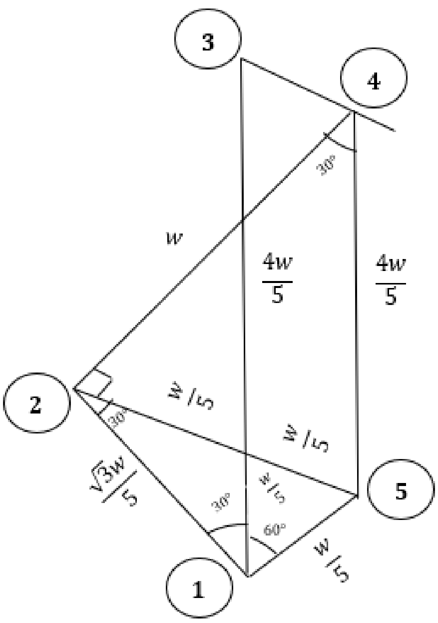
$T_0 = 6W$ (5)

15

(**b**) $A \circlearrowleft \quad P.\frac{5a}{2} - Wa\sin 60^0 = 0$ (5)

$P = \frac{\sqrt{3}W}{5}$ (5)

Rod	*Stress*	*Magnitute*	
AB	*Tension*	$W/5$	10
BC	*Tension*	$W/5$	10
CD	*Thrust*	$2W/5$	10
AD	*Thrust*	$2\sqrt{3}W/5$	10
BD	*Tension*	$4W/5$	10



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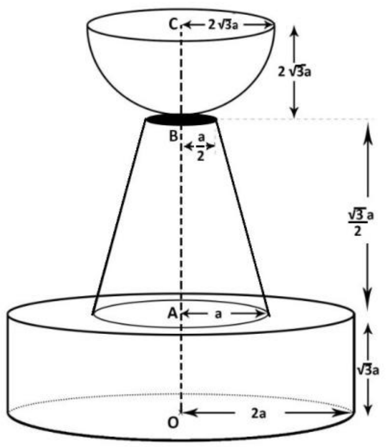
16. *(a)* Show that the centre of mass of

(i) A thin uniform hemispherical shell of radius a is at a distance $\frac{a}{2}$ from its centre.

(ii) A uniform right circular cone of height h is at a distance $\frac{h}{3}$ from the centre of the base of the cone.

(b) An Olympic lamp is made by rigidly fixing a hemispherical shell on the top of a uniform frustum of a hollow right circular cone of radii $\frac{a}{2}$ and a and of height $\frac{\sqrt{3}\,a}{2}$. The frustum rigidly stands on the top surface of a uniform solid cylinder as shown in the adjoining figure as follows.



- Uniform hemispherical shell is of radius $2\sqrt{3}a$ and centre C.
- Solid cylinder is of radius $2a$ and height $\sqrt{3}$a.
- The mass per unit volume of the solid cylinder is $\frac{2\sigma}{\pi}$, and the mass per unit area of the frustum and hemispherical shell is $\frac{a\sigma}{\pi}$
- Assume that the total mass of the Olympic lamp does not change due to the welding.
- The top B of the frustum is closed with a **light** circular disk of the same radius.

Show that the distance from O to the centre of mass of the Olympic lamp is $\left(\frac{24\sqrt{3}+371}{48+51\sqrt{3}}\right)a$.

Consider $\left(\frac{24\sqrt{3}+371}{48+51\sqrt{3}}\right)a \approx 3a$.

The Olympic lamp stands on a rough inclined plane. Assuming that the contact is rough enough to prevent from slipping find the inclination of the plane when the Olympic lamp is about to topple.

(a) (i) By symmetry the centre of mass G lies on the x-axis

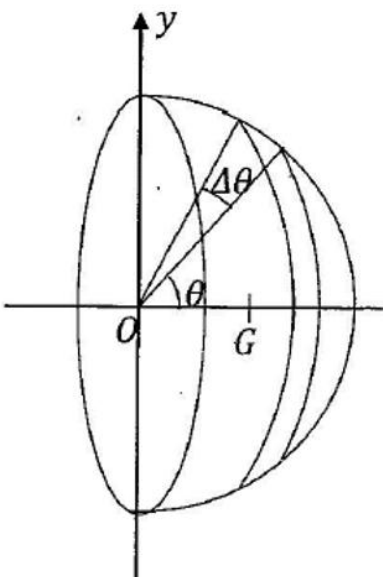
$\Delta m = 2\pi(a\sin\theta)a\rho\theta$, where σ is the mass per unit area (5)

Let $OG = \bar{x}$

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} 2\pi(a\sin\theta)a\sigma a\cos\theta\, d\theta}{\int_0^{\frac{\pi}{2}} 2\pi(a\sin\theta)a\sigma d\theta} \quad (5) + (5)$$

$$= \frac{[a\sin\theta]_0^{\frac{\pi}{2}}}{[-\cos\theta]_0^{\frac{\pi}{2}}} \quad (5) + (5)$$

$$= \frac{a}{2} \quad (5)$$



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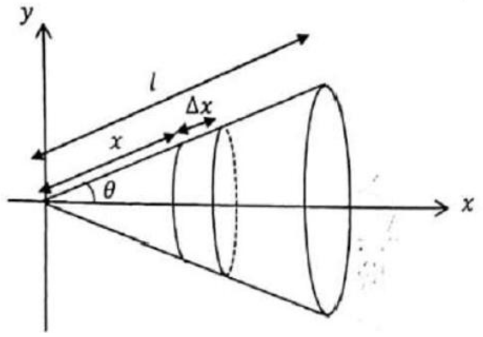
(ii) By symmetry the centre of mass G lies on the x-axis



$h = l\cos\theta$ (5)

$\Delta m = 2\pi(x\sin\theta)\Delta x\sigma$, where σ is the mass per unit length

$$\bar{x} = \frac{\int_0^l x\cos\theta\, 2\pi\sigma x\sin\theta\, dx}{\int_0^l 2\pi\sigma x\sin\theta\, dx}$$ (5) + (5)

$$= \frac{\cos\theta \int_0^l x^2 dx}{\int_0^l x dx}$$

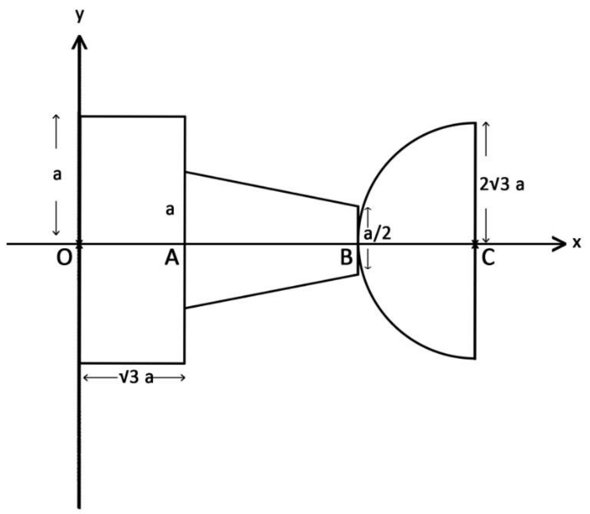
$$= \frac{\left[\frac{h}{2}\frac{x^2}{3}\right]_0^l}{\left[\frac{x^2}{2}\right]_0^l}$$ (5)

$$= \frac{2h}{3}$$ (5)

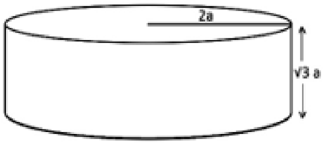
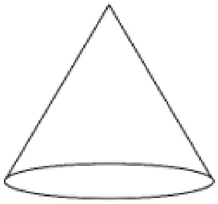
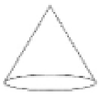
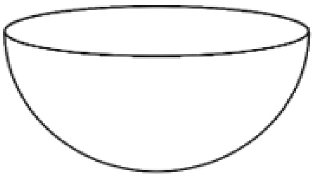
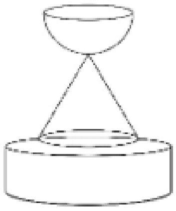
∴ The required distance $= \frac{h}{3}$

25

(b)



According to the symmetry C.O.G lies on the X axis. (5)

OBJECT	MASS	DISTANCE FROM O to C.O.G
	$8\sqrt{3}a^3\sigma$ (5)	$\frac{\sqrt{3}a}{2}$ (5)
	$2a^3\sigma$ (5)	$(\frac{a}{\sqrt{3}} + \sqrt{3}a) = \frac{4a}{\sqrt{3}}$ (5)
	$\frac{a^3\sigma}{2}$ (5)	$\sqrt{3}a + \frac{\sqrt{3}}{2}a + \frac{a}{2\sqrt{3}} = \frac{5a}{\sqrt{3}}$ (5)
	$24a^3\sigma$ (5)	$\frac{5\sqrt{3}a}{2}$ (5)
	$(\frac{16\sqrt{3}+51}{2})a^3\sigma$ (5)	$\bar{X}$

let $a^3\sigma=\lambda$

Taking

moments around Y axis

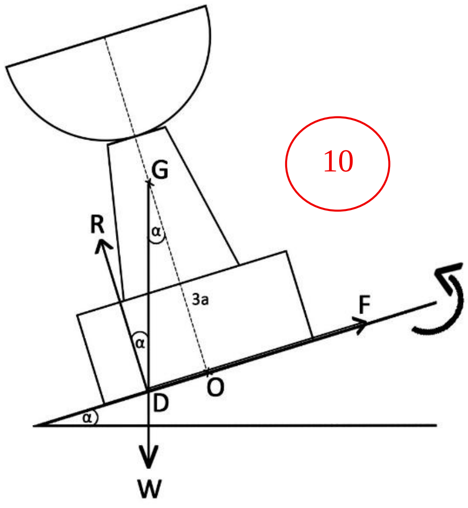
$(\frac{16\sqrt{3}+51}{2})\lambda\bar{X} = 8\sqrt{3} \times \frac{\sqrt{3}a}{2}\lambda + 2 \times \frac{4a}{\sqrt{3}}\lambda - \frac{\lambda}{2} \cdot \frac{5a}{\sqrt{3}} + 24\lambda\frac{5\sqrt{3}}{2}a$ (10)

$(\frac{16\sqrt{3}+51}{2})\bar{X} = 12a + \frac{8a}{\sqrt{3}} - \frac{5a}{2\sqrt{3}a} + 60\sqrt{3}a$

$(\frac{16\sqrt{3}+51}{2})\bar{X} = \frac{(24\sqrt{3}+371)a}{2\sqrt{3}}$ (5)

$\bar{X} = \frac{(24\sqrt{3}+371)}{16\sqrt{3}+51} \cdot \frac{a}{\sqrt{3}}$

$\bar{X} = \frac{(24\sqrt{3}+371)}{48+51\sqrt{3}} \simeq 3.02a$



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ODG△

$\tan\propto = \frac{OD}{3a}$

$OD = 3a \tan\propto$

Olympic lamp is about to topple

$OD < 2a$

10

$3a \tan\alpha < 2a$

$\tan\alpha < \frac{2}{3}$

$\propto < \tan^{-1}(\frac{2}{3})$

When Olympic lamp is about to topple

$OD = 2a$

$3a \tan\alpha = 2a$

$\tan\alpha = \frac{2}{3}$

10

$\propto = \tan^{-1}(\frac{2}{3})$

17. (a) A box contains 4 red balls and 6 blue balls which are equal in all respects, except for colour. Two balls are taken out twice at random from the box, one after the other.

The experiment is conducted as follows:

- A fair die is rolled, and if the die shows an even number, the balls are taken out with replacement.
- If the die shows an odd number, the balls are taken out without replacement.

(i) Find the probability that both balls drawn are **red**.

(ii) If it is observed that both balls taken out are red, find the probability that the selection was made **without replacement**.

(b) Consider the following frequency distribution with k class intervals, where x_i is the class mark and f_i is the frequency of the i^{th} class interval, respectively for $i = 1,2,3 \ldots k$

Midpoint of the class interval	Frequency
x_1	f_1
x_2	f_2
…..	….
x_i	f_i
……	……
x_k	f_k

Write down the mean and the standard deviation of the above frequency distribution.

The deposits of 100 customers of a micro savings and loan community club over five years are listed below in the table.

Deposit in 10,000 rupees	Number of members
$15 - 19$	10
$20 - 24$	20
$25 - 29$	f
$30 - 34$	g
$35 - 39$	p
$40 - 44$	7
Total	100

It is given that frequency of the modal class is 28, the frequency of the median class is 21 and Median $<$ Mode$<$ 34.5, find f, g and p, justifying your answers.

(i) Calculate the median deposit of a member in rupees.

(ii) Using an assumed mean of 320,000 rupees, calculate the mean deposit of a member in rupees.

(iii) Show that the standard deviation of the deposits, correct to one decimal place, is 69,500 rupees.

(iv) Find the number of members who have deposited less than 395,000 rupees.

(a) (i). A – Event of both balls drawn being red

E – Event of the die showing an even number

O – Event of the die showing an odd number

$P(E) = P(O) = \frac{1}{2}$ (5)

$P(A/E) = \frac{4}{10} \times \frac{4}{10}$ (10)

$P(A/O) = \frac{4}{10} \times \frac{3}{9}$ (10)

$P(A) = P(E) \times P(A/E) + P(O) \times P(A/O)$

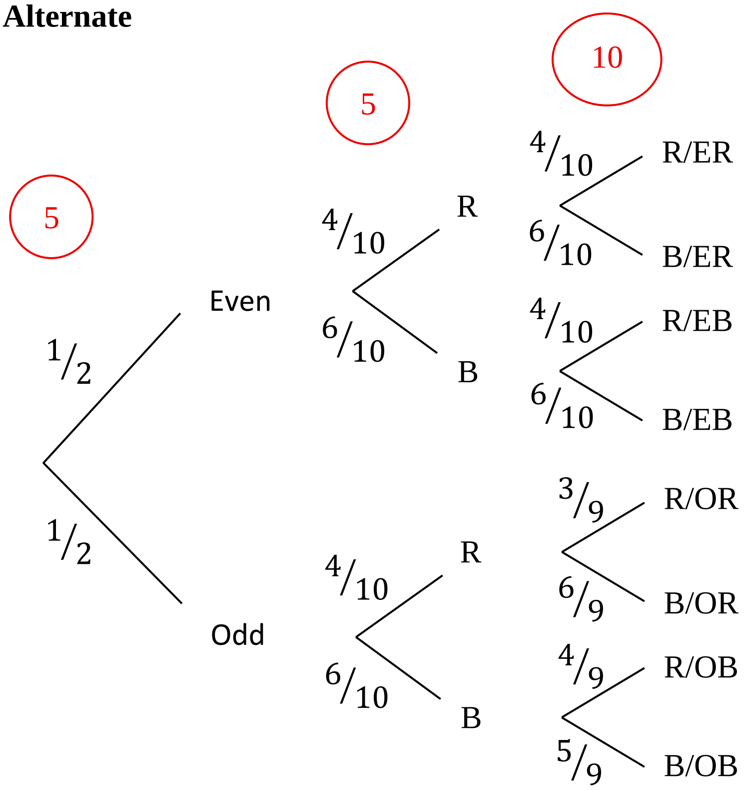
$= \frac{1}{2} \times \frac{4}{10} \times \frac{4}{10} + \frac{1}{2} \times \frac{4}{10} \times \frac{3}{9}$

(5) (5)

$= \frac{2}{25} + \frac{1}{15} = \frac{11}{75}$ (5)

40

Alternate



Probability that both balls are red $= \frac{1}{2} \times \frac{4}{10} \times \frac{4}{10} + \frac{1}{2} \times \frac{4}{10} \times \frac{3}{9}$ (5) (5) For the equation (5)

$= \frac{2}{25} + \frac{1}{15} = \frac{11}{75}$ (5)

(ii) Using Bayes' Theorem;

$$P(O/R) = \frac{P(O)\,.\,P(R/O)}{P(R)}$$ (5)

$$= \frac{\frac{1}{2} \cdot \frac{4}{10} \cdot \frac{3}{9}}{\frac{11}{75}} = \frac{1}{15} \times \frac{75}{11}$$ (5) (5)

$$= \frac{5}{11}$$ (5)

20

(b).

$class\ mark$	$frequency$	$f_i x_i$	$f_i {x_i}^2$
x_1	f_1	$f_1 x_1$	$f_1 {x_1}^2$
x_2	f_2	$f_2 x_2$	$f_2 {x_2}^2$
……	….	….	….
x_i	f_i	$f_i x_i$	$f_i {x_i}^2$
….	….	….	….
x_t	f_k	$f_k x_k$	$f_k {x_k}^2$
	$\sum_{i=1}^{n} f_i$	$\sum_{i=1}^{n} f_i x_i$	$\sum_{i=1}^{n} f_i {x_i}^2$

$$Mean\ \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$ (5)

$$\textbf{standard deviation} = \boldsymbol{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} f_i {x_i}^2}{n} - \bar{x}^2}$$ (5)

10

5 5 5 5

Class boundaries 10,000	Class mark x_i	Frequency f_i	Cumulative frequency F_i	u_i	$f_i u_i$	U_i^2	$f_i U_i^2$
14.5-19.5	17	10	10	-3	-30	9	90
19.5-24.5	22	20	30	-2	-40	4	80
24.5-29.5	27	21	51	-1	-21	1	21
29.5-34.5	**32**	28	79	0	0	0	0
34.5-39.5	37	14	93	1	14	1	14
39.5-44.5	42	7	100	2	14	4	28
		100			-63		233

Let $u_i = \frac{x_i - A}{C}$ where A is the assumed mean and C is the class width.

Median class = class in which the $\left(\frac{100}{2}\right)^{th}$ observation included = 24.5-29.5

$$\therefore f = 28 \longrightarrow (1)$$ 5

$$medal\ class = 29.5 - 34.5$$

$$since\ \ median\ < \text{Mode} < 34.5$$ 5

$$\therefore g = 28 \longrightarrow (2)$$ 5

$$\therefore p = 100 - (10 + 30 + 7 + 28 + 21)$$

$$\therefore p = 14$$ 5

(i). $median\ Deposit = (24.5 + \frac{5}{21} \times 20) \times 10{,}000$ 5

$median\ deposit = (29.26) \times 10{,}000$

$= 293{,}000\ rupees$ 5

(ii). $Mean = \left[32 + 5 \times \left(\frac{-63}{100}\right)\right] \times 10{,}000$ 5

$= 288{,}500$ 5

(iii). $\text{standard deviation} = \sqrt{25\left[\left(\frac{233}{100}\right) - \left(\frac{63}{100}\right)^2\right]}$ 5

$= \sqrt{48.3275} = 6.95$ 5

$= 6.95 \times 10{,}000$

$= Rs\ 69{,}500$ 5

(iv). 93 5

80

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